Unification theorems in algebraic geometry

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Abstract

In this paper, for a given finitely generated algebra (an algebraic structure with arbitrary operations and no predicates) $\mathcal{A}$ we study finitely generated limit algebras of $\mathcal{A}$, approaching them via model theory and algebraic geometry. Along the way we lay down foundations of algebraic geometry over arbitrary algebraic structures.

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1 Introduction

Quite often relations between sets of elements of a fixed algebraic structure $\mathcal{A}$ can be described in terms of equations over $\mathcal{A}$. In the classical case, when $\mathcal{A}$ is a field, the area of mathematics where such relations are studied is known under the name of *algebraic geometry*. It is natural to use the same name in the general case. Algebraic geometry over arbitrary algebraic structures is a new area of research in modern algebra, nevertheless, there are already several breakthrough particular results here, as well as, interesting developments of a general theory. Research in this area started with a series of papers by Plotkin [36, 37], Baumslag, Kharlampovich, Myasnikov, and Remeslennikov [4, 34, 24, 25].

There are general results which hold in the algebraic geometries over arbitrary algebraic structures, we refer to them as the *universal algebraic geometry*. The main purpose of this paper is to lay down the basics of the universal algebraic geometry in a coherent form. We emphasize here the relations between model theory, universal algebra, and algebraic geometry. Another goal is quite pragmatic — we intend to unify here some common methods known in different fields under different names. Also, there are several essentially the same results that independently occur in various branches of modern algebra, were they are treated by means specific to the area. Here we give very general proofs of these results based on model theory and universal algebra.

Limit algebras, in all their various incarnations, are the main object of this paper. The original notion came from group theory where limit groups play a prominent part. The limit groups of a fixed group G appear in many different situations: in combinatorial group theory as groups discriminated by G (ω-residually G-groups or fully residually G-groups) [2, 3, 34, 5, 6], in the algebraic geometry over groups as the coordinate groups of irreducible varieties over G [4, 24, 25, 26, 47], groups universally equivalent to G [40, 13, 34], limit groups of G in the Grigorchuk-Gromov's metric [10], in the theory of equations in groups [28, 38, 39, 24, 25, 26, 18], in group actions [8, 12, 35, 17, 14], in the solutions of Tarski problems [27, 48], etc. These numerous characterizations of limit groups make them into a very robust tool linking group theory, topology and logic. It turned out that many of the results on limit groups can be naturally generalized to Lie algebras [19, 20, 21, 22, 23].

Our prime objective is to convey some basic facts of the general theory of limit algebras in an arbitrary language. We prove the so-called unification theorems for limit groups that show that the characterization results above hold in the general case as well.

2 Preliminaries

2.1 Languages and structures

Let $\mathcal{L} = \mathcal{F} \cup \mathcal{P} \cup \mathcal{C}$ be a first-order *language* (or a *signature*), consisting of a set $\mathcal{F}$ of symbols of operations F (given together with their arities n_F), a set $\mathcal{P}$ of symbols of predicates P (given together with their arities n_P) and a set of constants $\mathcal{C}$. If $\mathcal{P} = \emptyset$ then the language $\mathcal{L}$ is *functional*, whereas $\mathcal{L}$ is *relational* if $\mathcal{F} = \mathcal{C} = \emptyset$.

For languages $\mathcal{L}_1 \subseteq \mathcal{L}_2$ we say that $\mathcal{L}_1$ is a *reduct* of $\mathcal{L}_2$ and $\mathcal{L}_2$ is an *expansion* of $\mathcal{L}_1$. The language $\mathcal{L}^{fun} = \mathcal{L} \smallsetminus \mathcal{P}$ is the functional part of $\mathcal{L}$. From now we fix a first-order functional language $\mathcal{L}$. Almost everything we prove holds (under appropriate adjustments) for arbitrary languages, but the exposition for functional languages is shorter.

Example 2.1. The language of *groups* consists of a binary operation $\cdot$ (multiplication), a unary operation $^{-1}$ (inversion), and a constant symbol e or 1 (the identity).

Example 2.2. The language of *unitary rings* consists of three binary operations $+$, $-$ and $\cdot$ (addition, subtraction and multiplication), and constants 0 and 1.

An $\mathcal{L}$-*structure* $\mathcal{M}$ is given by the following data: (i) a non-empty set M called the *universe* of $\mathcal{M}$; (ii) a function $F^{\mathcal{M}} : M^{n_F} \to M$ of arity n_F for each $F \in \mathcal{F}$; (iii) an element $c^{\mathcal{M}} \in M$ for each $c \in \mathcal{C}$. We often white the structure as $\mathcal{M} = \langle M; F^{\mathcal{M}}, c^{\mathcal{M}}, F \in \mathcal{F}, c \in \mathcal{C} \rangle$. We refer to $F^{\mathcal{M}}$ and $c^{\mathcal{M}}$ as *interpretations* of the symbols F and c in $\mathcal{M}$, and sometimes omit superscripts $\mathcal{M}$ (when the interpretation is obvious from the context). Typically we denote structures in $\mathcal{L}$ by capital calligraphic letters and their universes (the underlying sets) by the corresponding capital Latin letters. Structures in a functional language are termed *algebras* (or *universal algebras*). An algebra $\mathcal{E}$ with the universe consisting of a single element is called *trivial*. Obviously, interpretation of symbols from $\mathcal{L}$ in $\mathcal{E}$ is unique.

As usual, one can define the notion of a homomorphism, and all its variations, between structures in a given language. If a subset $N \subseteq M$ is closed under the operations F of $\mathcal{F}$ and contains all the constants $c \in \mathcal{C}$ then restrictions of the operations F onto N, together with the constants c, determine a new $\mathcal{L}$-structure, called a *substructure* $\mathcal{N}$ of $\mathcal{M}$, in which case we write $\mathcal{N} \leq \mathcal{M}$. For a subset $M' \subseteq M$ the intersection of all substructures of $\mathcal{M}$ containing M' is a substructure $\mathcal{M}'$ of $\mathcal{M}$ generated by M' (so M' is a generating set for $\mathcal{M}'$), symbolically $\mathcal{M}' = \langle M' \rangle$. $\mathcal{M}$ is termed *finitely generated* if it has a finite generating set.

Let $X = \{x_1, x_2, \ldots\}$ be a finite or countable set of variables. *Terms* in $\mathcal{L}$ in variables X are formal expressions defined recursively as follows:

T1) variables $x_1, x_2, \ldots, x_n, \ldots$ are terms;

T2) constants from $\mathcal{L}$ are terms;

T3) if $F(x_1, \ldots, x_n) \in \mathcal{F}$ and $t_1, \ldots, t_n$ are terms then $F(t_1, \ldots, t_n)$ is a term.

For $F \in \mathcal{F}$ we write $F(x_1, \ldots, x_n)$ to indicate that $n = n_F$.

By $\mathrm{T}_{\mathcal{L}} = \mathrm{T}_{\mathcal{L}}(X)$ we denote the set of all terms in $\mathcal{L}$. For a term $t \in \mathrm{T}_{\mathcal{L}}$ one can define the set of variables $V(t) \subset X$ that occur in t. We write $t(x_1, \ldots, x_n)$ to indicate that $V(t) \subseteq \{x_1, \ldots, x_n\}$. Also, we use the vector notation $t(\bar{x})$, where $\bar{x} = (x_1, \ldots, x_n)$. Following the recursive definition of t one can define in a natural way a function $t^{\mathcal{M}} : M^n \to M$ (which we sometimes again denote by t). If $V(t) = \emptyset$ then t is a *closed* term and $t^{\mathcal{M}}$ is just a constant. Observe, that the universe of the substructure of $\mathcal{M}$ generated by a subset $M' \subseteq M$ is equal to $\bigcup\{t(M') \mid t \in \mathrm{T}_{\mathcal{L}}(X)\}$, where $t(M')$ is the range of the function t.

The condition T3) allows one to define an operation $F^{\mathcal{T}_{\mathcal{L}}(X)}$ on the set of terms $\mathrm{T}_{\mathcal{L}}(X)$. By T2) the set $\mathrm{T}_{\mathcal{L}}(X)$ contains all constants from $\mathcal{L}$, which gives a natural interpretation of constants in $\mathrm{T}_{\mathcal{L}}(X)$. These altogether turn the set $\mathrm{T}_{\mathcal{L}}(X)$ into an $\mathcal{L}$-structure $\mathcal{T}_{\mathcal{L}}(X)$, which is called the *absolutely free* $\mathcal{L}$-algebra with basis X. The name comes from the the following universal property of $\mathcal{T}_{\mathcal{L}}(X)$: for any $\mathcal{L}$-structure $\mathcal{M}$ a map $h : X \to M$, extends to a unique $\mathcal{L}$-homomorphism $h : \mathcal{T}_{\mathcal{L}}(X) \to \mathcal{M}$.

Formulas in $\mathcal{L}$ (in variables X) are defined recursively as follows:

F1) if $t, s \in T_{\mathcal{L}}(X)$ then $(t = s)$ is a formula (called an *atomic formula*);

F2) if ϕ and ψ are formulas then $\neg\phi, (\phi \vee \psi), (\phi \wedge \psi), (\phi \to \psi)$ are formulas;

F3) If ϕ is a formula and x is a variable then $\forall x\phi$ and $\exists x\phi$ are formulas.

For a formula ϕ one can define the set $V(\phi)$ of *free variables* of ϕ according to the rules F1)–F3). Namely, $V(t_1 = t_2) = V(t_1) \cup V(t_2)$, $V(\neg\phi) = V(\phi)$, $V(\phi \circ \psi) = V(\phi) \cup V(\psi)$, where $\circ \in \{\vee, \wedge, \to\}$, and $V(\forall x\phi) = V(\exists x\phi) = V(\phi) \smallsetminus \{x\}$. We write $\phi(x_1, \ldots, x_n)$ in the case when $V(\phi) \subseteq \{x_1, \ldots, x_n\}$. Let $\Phi_{\mathcal{L}}(X)$ be the set of all formulas in $\mathcal{L}$ with $V(\phi) \subseteq X$. A formula ϕ with $V(\phi) = \emptyset$ termed a *sentence*, or a *closed* formula.

If $\phi(x_1, \ldots, x_n) \in \Phi_{\mathcal{L}}(X)$ and $m_1, \ldots, m_n \in M$ then one can define, following the conditions F1)–F3), the relation "ϕ is true in $\mathcal{M}$ under the interpretation $x_1 \to m_1, \ldots, x_n \to m_n$" (symbolically $\mathcal{M} \models \phi(m_1, \ldots, m_n)$). It is convenient sometimes to view this relation as an n-ary predicate ϕ^M on M. If $h : X \to M$ is an interpretation of variables then we denote $\phi^h = \phi^M(h(x_1), \ldots, h(x_n))$.

A set of formulas $\Phi \subseteq \Phi_{\mathcal{L}}(X)$ is *consistent* if there is an $\mathcal{L}$-structure $\mathcal{M}$ and an interpretation $h : X \to M$ such that $\mathcal{M} \models \phi^h$ for every $\phi \in \Phi$. In this case one says that Φ is *realized* in $\mathcal{M}$.

The following result is due to Malcev, it plays a crucial role in model theory.

Theorem [Compactness Theorem] *Let* $\mathbf{K}$ *be a class of* $\mathcal{L}$*-structures and* $\Phi \subseteq \Phi_{\mathcal{L}}(X)$*. If every finite subset of* Φ *is realized in some structure in* $\mathbf{K}$ *then the whole set* Φ *is realized in some ultraproduct of structures from* $\mathbf{K}$*.*

2.2 Theories

Two formulas $\phi, \psi \in \Phi_{\mathcal{L}}(X)$ are called *equivalent* if $\phi^h = \psi^h$ for any interpretation $h : X \to M$ and any $\mathcal{L}$-structure $\mathcal{M}$. One of the principle results in mathematical logic states that any formula $\phi \in \Phi_{\mathcal{L}}(X)$ is equivalent to a formula ψ in the following form:

$$Q_1 x_1 \ldots Q_m x_m \left(\bigvee_{i=1}^{n} (\bigwedge_{j=1}^{k} \psi_{ij}) \right), \tag{1}$$

where $Q_i \in \{\forall, \exists\}$ and ψ_{ij} is an atomic formula or its negation. One of the standard ways to characterize complexity of formulas is according to their quantifier prefix $Q_1 x_1 \ldots Q_m x_m$ in (1).

If in (1) all the quantifiers Q_i are universal then the formula ψ is called *universal* or $\forall$-formula, and if all of them are existential then ψ is *existential* or $\exists$-formula. In this fashion ψ is $\forall\exists$-formula if the prefix has only one alteration of quantifiers (from $\forall$ to $\exists$). Similarly, one can define $\exists\forall$-formulas. Observe, that $\forall$- and $\exists$-formulas are dual relative to negation, i.e., the negation of $\forall$-formula is equivalent to an $\exists$-formula, and the negation of $\exists$-formula is equivalent to an $\forall$-formula. A similar result holds for $\forall\exists$- and $\exists\forall$-formulas. One may consider formulas with more alterations of quantifiers, but we have no use of them in this paper.

A formula in the form (1) is *positive* if it does not contain negations (i.e., all ψ_{ij} are atomic). A formula is *quantifier-free* if it does not contain quantifiers. We denote the set of all quantifier-free formulas from $\Phi_{\mathcal{L}}(X)$ by $\Phi_{\mathrm{qf},\mathcal{L}}(X)$, and the set of all atomic formulas by $\mathrm{At}_{\mathcal{L}}(X)$.

Recall that a *theory* in the language $\mathcal{L}$ is an arbitrary consistent set of sentences in $\mathcal{L}$. A theory T is *complete* if for every sentence ϕ either ϕ or $\neg\phi$ lies in T. By $\mathrm{Mod}(T)$ we denote the (non-empty) class of all $\mathcal{L}$-structures $\mathcal{M}$ which satisfy all the sentences from T. Structures from $\mathrm{Mod}(T)$ are termed *models* of T and T is a set of *axioms* for the class $\mathrm{Mod}(T)$. Conversely, if $\mathbf{K}$ is a class of $\mathcal{L}$-structures then the set $\mathrm{Th}(\mathbf{K})$ of sentences, which are true in all structures from $\mathbf{K}$, is called the *elementary theory* of $\mathbf{K}$. Similarly, the set $\mathrm{Th}_{\forall}(\mathbf{K})$ ($\mathrm{Th}_{\exists}(\mathbf{K})$) of all $\forall$-sentences ($\exists$-sentences) from $\mathrm{Th}(\mathbf{K})$ is called the *universal* (*existential*) theory of $\mathbf{K}$. The following notions play an important part in this paper. Two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$ are *elementarily equivalent* if $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$, and they are *universally* (*existentially*) equivalent if $\mathrm{Th}_{\forall}(\mathcal{M}) = \mathrm{Th}_{\forall}(\mathcal{N})$ ($\mathrm{Th}_{\exists}(\mathcal{M}) = \mathrm{Th}_{\exists}(\mathcal{N})$). In this event we write, correspondingly, $\mathcal{M} \equiv \mathcal{N}$, $\mathcal{M} \equiv_{\forall} \mathcal{N}$ or $\mathcal{M} \equiv_{\exists} \mathcal{N}$. Notice, that due to the duality mentioned above $\mathcal{M} \equiv_{\forall} \mathcal{N} \Longleftrightarrow \mathcal{M} \equiv_{\exists} \mathcal{N}$ for arbitrary $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$.

A class of $\mathcal{L}$-structures $\mathbf{K}$ is *axiomatizable* if $\mathbf{K} = \mathrm{Mod}(T)$ for some theory T in $\mathcal{L}$. In particular, $\mathbf{K}$ is $\forall$- ($\exists$-, or $\forall\exists$-) axiomatizable if the theory T is $\forall$- ($\exists$-, or $\forall\exists$-) theory.

3 Algebras

There are several types of classes of $\mathcal{L}$-structures that play a part in general algebraic geometry: prevariaeties, quasivarieties, universal closures, and $\mathcal{A}$-algebras. We refer to [34] for a detailed discussion on this and related matters. Here we present only a few properties and characterizations of these classes, that will be used in the sequel. Most of them are known and can be found in the classical books on universal algebra, for example, in [30]. On the algebraic theory of quasivarieties, the main subject of this section, we refer to [15].

3.1 Congruences

In this section we remind some notions and introduce notation on presentation of algebras via generators and relations.

Let $\mathcal{M}$ be an arbitrary fixed $\mathcal{L}$-structure. An equivalence relation θ on M is a *congruence* on $\mathcal{M}$ if for every operation $F \in \mathcal{F}$ and any elements $m_1, \ldots, m_{n_F}, m'_1, \ldots, m'_{n_F} \in M$ such that $m_i \sim_\theta m'_i$, $i = 1, \ldots, n_F$, one has $F^{\mathcal{M}}(m_1, \ldots, m_{n_F}) \sim_\theta F^{\mathcal{M}}(m'_1, \ldots, m'_{n_F})$.

For a congruence θ the operations $F^{\mathcal{M}}$, $F \in \mathcal{F}$, naturally induce well-defined operations on the factor-set M/θ. Namely, if we denote by m/θ the equivalence class of $m \in M$ then $F^{\mathcal{M}/\theta}$ is defined by

$$F^{\mathcal{M}/\theta}(m_1/\theta, \ldots, m_{n_F}/\theta) = F^{\mathcal{M}}(m_1, \ldots, m_{n_F})/\theta$$

for any $m_1, \ldots, m_{n_F} \in M$. Similarly, $c^{\mathcal{M}/\theta}$ is defined for $c \in \mathcal{C}$ as the class $c^{\mathcal{M}}/\theta$. This turns the factor-set $\mathcal{M}/\theta$ into an $\mathcal{L}$-structure. It follows immediately from the construction that the map $h : M \to M/\theta$, such that $h(m) = m/\theta$, is an $\mathcal{L}$-epimorphism $h : \mathcal{M} \to \mathcal{M}/\theta$, called the *canonical* epimorphism.

The set $\mathrm{Con}(\mathcal{M})$ of all congruences on $\mathcal{M}$ forms a lattice relative to the inclusion $\theta_1 \leqslant \theta_2$, i.e., every two congruences in $\mathrm{Con}(\mathcal{M})$ have the least upper and the greatest lower bounds in the ordered set $\langle \mathrm{Con}(\mathcal{M}), \leqslant \rangle$. To see this, observe first that the intersection of an arbitrary set $\Theta = \{\theta_i, i \in I\}$ of congruences on $\mathcal{M}$ is again a congruence on $\mathcal{M}$, hence the greatest lower bound for Θ. Now, the intersection of the non-empty set $\{\theta \in \mathrm{Con}(\mathcal{M}) \mid \theta_i \leqslant \theta \ \forall\, \theta_i \in \Theta\}$ is the least upper bound for Θ. The following result is easy.

Lemma 3.1. *Let $\mathcal{M}$ be an $\mathcal{L}$-algebra, $\{\theta_i \mid i \in I\} \subseteq \mathrm{Con}(\mathcal{M})$ and $\theta = \bigcap_{i \in I} \theta_i$. Then $\mathcal{M}/\theta$ embeds into the direct product $\prod_{i \in I} \mathcal{M}/\theta_i$ via the diagonal monomorphism $m/\theta \to \prod_{i \in I} m/\theta_i$.*

A homomorphism $h : \mathcal{M} \to \mathcal{N}$ of two $\mathcal{L}$-structures determines the *kernel* congruence $\ker h$ on $\mathcal{M}$, which is defined by

$$m_1 \sim_{\ker h} m_2 \quad \Longleftrightarrow \quad h(m_1) = h(m_2), \quad m_1, m_2 \in M.$$

Observe, that if $\theta \in \mathrm{Con}(\mathcal{M})$ and $\theta \leqslant \ker h$ then the map $\bar{h} : \mathcal{M}/\theta \to \mathcal{N}$ defined by $\bar{h}(m/\theta) = h(m)$ for $m \in M$ is a homomorphism of $\mathcal{L}$-structures.

Definition 3.2. A set of atomic formulas $\Delta \subseteq \mathrm{At}_{\mathcal{L}}(X)$ is called *congruent* if the binary relation θ_Δ on the set of terms $\mathrm{T}_{\mathcal{L}}(X)$ defined by (where $t_1, t_2 \in \mathrm{T}_{\mathcal{L}}(X)$)

$$t_1 \sim_{\theta_\Delta} t_2 \quad \Longleftrightarrow \quad (t_1 = t_2) \in \Delta.$$

is a congruence on the free $\mathcal{L}$-algebra $\mathcal{T}_{\mathcal{L}}(X)$.

The following lemma characterizes congruent sets of formulas.

Lemma 3.3. *A set of atomic formulas $\Delta \subseteq \mathrm{At}_{\mathcal{L}}(X)$ is congruent if and only if it satisfies the following conditions:*

1. $(t = t) \in \Delta$ *for any term* $t \in \mathrm{T}_{\mathcal{L}}(X)$*;*
2. *if* $(t_1 = t_2) \in \Delta$ *then* $(t_2 = t_1) \in \Delta$ *for any terms* $t_1, t_2 \in \mathrm{T}_{\mathcal{L}}(X)$*;*
3. *if* $(t_1 = t_2) \in \Delta$ *and* $(t_2 = t_3) \in \Delta$ *then* $(t_1 = t_3) \in \Delta$ *for any terms* $t_1, t_2, t_3 \in \mathrm{T}_{\mathcal{L}}(X)$*;*
4. *if* $(t_1 = s_1), \ldots, (t_{n_F} = s_{n_F}) \in \Delta$ *then* $(F(t_1, \ldots, t_{n_F}) = F(s_1, \ldots, s_{n_F})) \in \Delta$ *for any terms* $t_i, s_i \in \mathrm{T}_{\mathcal{L}}(X)$, $i = 1, \ldots, n_F$, *and any functional symbol* $F \in \mathcal{L}$.

Proof. Straightforward. $\square$

Since the intersection of an arbitrary set of congruent sets of atomic formulas is again congruent, it follows that for a set $\Delta \subseteq \mathrm{At}_{\mathcal{L}}(X)$ there is the least congruent subset $[\Delta] \subseteq \mathrm{At}_{\mathcal{L}}(X)$, containing Δ. Therefore, Δ uniquely determines the congruence $\theta_\Delta = \theta_{[\Delta]}$.

For an $\mathcal{L}$-algebra $\mathcal{M}$ generated by a set $M' \subseteq M$ put $X = \{x_m \mid m \in M'\}$ and consider a set $\Delta_{M'}$ of all atomic formulas $(t_1 = t_2) \in \mathrm{At}_{\mathcal{L}}(X)$ such that $\mathcal{M} \models (t_1 = t_2)$ under the interpretation $x_m \to m, m \in M'$. Obviously, $\Delta_{M'}$ is a congruent set in $\mathrm{At}_{\mathcal{L}}(X)$ (the set of all relation in $\mathcal{M}$ relative to M'). A subset $S \subseteq \Delta_{M'}$ is called a set of *defining relations* of $\mathcal{M}$ relative to M' if $[S] = \Delta_{M'}$. In this event the pair $\langle X \mid S\rangle$ termed a *presentation* of $\mathcal{M}$ by generators X and relations S.

Lemma 3.4. *If $\langle X \mid S\rangle$ is a presentation of $\mathcal{M}$ then $\mathcal{M} \cong \mathcal{T}_{\mathcal{L}}(X)/\theta_S$.*

Proof. The map $h' : X \to M'$ defined by $h'(x_m) = m$, $m \in M'$, extends to a homomorphism $h : \mathcal{T}_{\mathcal{L}}(X) \to \mathcal{M}$. Clearly, $t_1 \sim_{\ker h} t_2$ if and only if $(t_1 = t_2) \in [S]$ for terms $t_1, t_2 \in \mathrm{T}_{\mathcal{L}}(X)$. Therefore, $\mathcal{T}_{\mathcal{L}}(X)/\theta_S \cong \mathcal{T}_{\mathcal{L}}(X)/\ker h$. Now the result follows from the isomorphism $\mathcal{T}_{\mathcal{L}}(X)/\ker h \cong \mathcal{M}$. $\square$

3.2 Quasivarieties

In this section we discuss quasivarieties and related objects. The main focus is on how to generate the least quasivariety containing a given class of structures $\mathbf{K}$. A model example here is the celebrated Birkhoff's theorem which describes $\mathbf{Var}(\mathbf{K})$, the smallest variety containing $\mathbf{K}$, as the class $\mathbf{HSP}(\mathbf{K})$ obtained from

$\mathbf{K}$ by taking direct products (the operator $\mathbf{P}$), then substructures (the operator $\mathbf{S}$), and then homomorphic images (the operator $\mathbf{H}$). Along the way we introduce some other relevant operators. On the algebraic theory of quasivarieties we refer to [15] and [30].

We fix, as before, a functional language $\mathcal{L}$ and a class of $\mathcal{L}$-algebras $\mathbf{K}$. We always assume that $\mathbf{K}$ is an abstract class, i.e., with any algebra $\mathcal{M} \in \mathbf{K}$ the class $\mathbf{K}$ contains all isomorphic copies of $\mathcal{M}$.

Recall that an *identity* in $\mathcal{L}$ is a formula of the type

$$\forall x_1 \ldots \forall x_n \left(t(x_1, \ldots, x_n) = s(x_1, \ldots, x_n)\right),$$

where t, s are terms in $\mathcal{L}$. Meanwhile, a *quasi-identity* is a formula of the type

$$\forall x_1 \ldots \forall x_n \left((\bigwedge_{i=1}^{m} t_i(\bar{x}) = s_i(\bar{x})) \quad \rightarrow \quad (t(\bar{x}) = s(\bar{x})) \right),$$

where $t(\bar{x}), s(\bar{x}), t_i(\bar{x}), s_i(\bar{x})$ are terms in $\mathcal{L}$ in variables $\bar{x} = (x_1, \ldots, x_n)$.

A class of $\mathcal{L}$-structures is called a *quasivariety* (*variety*) if it can be axiomatized by a set of quasi-identities (identities). Given a class of $\mathcal{L}$-structures $\mathbf{K}$ one can define the quasivariety $\mathbf{Qvar}(\mathbf{K})$, generated by $\mathbf{K}$, as the quasivariety axiomatized by the set $\mathrm{Th}_{\mathrm{qi}}(\mathbf{K})$ of all quasi-identities which are true in all structures from $\mathbf{K}$, i.e., $\mathbf{Qvar}(K) = \mathrm{Mod}(\mathrm{Th}_{\mathrm{qi}}(\mathbf{K}))$. Notice, that $\mathbf{Qvar}(\mathbf{K})$ is the least quasivariety containing $\mathbf{K}$. Similarly, one defines the variety $\mathbf{Var}(\mathbf{K})$ generated by $\mathbf{K}$.

Observe, that an identity $\forall\, \bar{x}(t(\bar{x}) = s(\bar{x}))$ is equivalent to a quasi-identity $\forall\, \bar{x}(x = x \rightarrow t(\bar{x}) = s(\bar{x}))$, therefore, $\mathbf{Qvar}(\mathbf{K}) \subseteq \mathbf{Var}(\mathbf{K})$.

Before we proceed with quasivarieties, we introduce one more class of structures. Namely, $\mathbf{K}$ termed a *prevariety* if $\mathbf{K} = \mathbf{SP}(\mathbf{K})$. By $\mathbf{Pvar}(\mathbf{K})$ we denote the least prevariety, containing $\mathbf{K}$. The prevariety $\mathbf{Pvar}(\mathbf{K})$ grasps the residual properties of the structures from $\mathbf{K}$. An $\mathcal{L}$-structure $\mathcal{M}$ is *separated* by $\mathbf{K}$ if for any pair of non-equal elements $m_1, m_2 \in M$ there is a structure $\mathcal{N} \in \mathbf{K}$ and a homomorphism $h : \mathcal{M} \rightarrow \mathcal{N}$ such that $h(m_1) \neq h(m_2)$. By $\mathbf{Res}(\mathbf{K})$ we denote the class of $\mathcal{L}$-structures separated by $\mathbf{K}$.

In the following lemma we collect some known facts on prevarieties.

Lemma 3.5. *For any class of $\mathcal{L}$-structures $\mathbf{K}$ the following holds:*

1) $\mathbf{Pvar}(\mathbf{K}) = \mathbf{SP}(\mathbf{K}) \subseteq \mathbf{Qvar}(\mathbf{K})$*;*

2) $\mathbf{Pvar}(\mathbf{K}) = \mathbf{Res}(\mathbf{K})$*;*

3) $\mathbf{Pvar}(\mathbf{K})$ *is axiomatizable if and only if* $\mathbf{Pvar}(\mathbf{K}) = \mathbf{Qvar}(\mathbf{K})$.

Proof. Equality 1) follows directly from definitions.

2) was proven for groups in [34], here we give a general argument. It is easy to see that $\mathbf{Res}(\mathbf{K})$ is a prevariety, so $\mathbf{Pvar}(\mathbf{K}) \subseteq \mathbf{Res}(\mathbf{K})$. To show converse, take a structure $\mathcal{M} \in \mathbf{Res}(\mathbf{K})$ and consider the set I of all pairs (m_1, m_2), $m_1, m_2 \in M$, such that $m_1 \neq m_2$. Then for every $i \in I$ there exists a structure

$\mathcal{N}_i \in \mathbf{K}$ and a homomorphism $h_i : \mathcal{M} \to \mathcal{N}_i$ with $h_i(m_1) \neq h_i(m_2)$. The homomorphisms $h_i, i \in I$, give rise to the "diagonal" homomorphism $h : \mathcal{M} \to \prod_{i\in I} \mathcal{N}_i$, which is injective by construction. Hence $\mathcal{M} \in \mathbf{SP}(\mathbf{K})$, as required.

3) is due to Malcev [31]. $\square$

Prevarieties play an important role in combinatorial algebra, they can be characterized as classes of structures admitting presentations by generators and relator. Namely, let X be a set and Δ a set of atomic formulas from $\Phi_{\mathcal{L}}(X)$. Following Malcev [30], we say that a presentation $\langle X \mid \Delta\rangle$ *defines a structure* $\mathcal{M}$ in a class $\mathbf{K}$ if there is a map $h : X \to M$ such that

D1) $h(X)$ generates $\mathcal{M}$ and all the formulas from Δ are realized in $\mathcal{M}$ under the interpretation h;

D2) for any structure $\mathcal{N} \in \mathbf{K}$ and any map $f : X \to N$ if all the formulas from Δ are realized in $\mathcal{N}$ under f then there exists a unique homomorphism $g : \mathcal{M} \to \mathcal{N}$ such that $g(h(x)) = f(x)$ for every $x \in X$.

If $\langle X \mid \Delta\rangle$ defines a structure in $\mathbf{K}$ then this structure is unique up to isomorphism, we denote it by $F_{\mathbf{K}}(X, \Delta)$.

Theorem [30] *A class* $\mathbf{K}$*, containing the trivial system* $\mathcal{E}$*, is a prevariety if and only if any presentation* $\langle X \mid \Delta\rangle$ *defines a structure in* $\mathbf{K}$.

To present similar characterizations for quasivarieties we need to introduce the following operators.

As was mentioned above, $\mathbf{P}(\mathbf{K})$ is the class of direct products of structures from $\mathbf{K}$. Recall, that the direct product of $\mathcal{L}$-structures $\mathcal{M}_i$, $i \in I$, is an $\mathcal{L}$-structure $\mathcal{M} = \prod_{i\in I} \mathcal{M}_i$ with the universe $M = \prod_{i\in I} M_i$ where the functions and constants from $\mathcal{L}$ are interpreted coordinate-wise. If all the structures $\mathcal{M}_i$ are isomorphic to some structure $\mathcal{N}$ then we refer to $\prod_{i\in I} \mathcal{M}_i$ as to a direct power of $\mathcal{N}$ and denote it by $\mathcal{N}^I$. By $\mathbf{P}_\omega(\mathbf{K})$ we denote the class of all finite direct products of structures from $\mathbf{K}$.

Recall, that a substructure $\mathcal{N}$ of a direct product $\prod_{i\in I} \mathcal{M}_i$ is a *subdirect product* of the structures $\mathcal{M}_i$, $i \in I$, if $p_j(\mathcal{N}) = \mathcal{M}_j$ for the canonical projections $p_j : \prod_{i\in I} \mathcal{M}_i \to M_j$, $j \in I$. By $\mathbf{P_s}(\mathbf{K})$ we denote the class of all subdirect products of structures from $\mathbf{K}$.

Let I be a set, D a filter over I (i.e., a collection D of subsets of I closed under finite intersections and such that if $a \in D$ then $b \in D$ for any $b \subseteq I$ with $a \subseteq b$, and also we assume that $\emptyset \notin D$), and $\{M_i \mid i \in I\}$ a family of sets. On the direct product $\prod_{i\in I} M_i$ one can define an equivalence relation $\sim_D$ such that $a \sim_D b$ if and only if $\{i \in I \mid p_i(a) = p_i(b)\} \in D$. We denote the factor-set by $\prod_{i\in I} M_i/D$, and the equivalence class of an element a by a/D. Now, if $\{\mathcal{M}_i \mid i \in I\}$ is a collection of $\mathcal{L}$-structures then the equivalence $\sim_D$ becomes a congruence on the direct product $\prod_{i\in I} \mathcal{M}_i$, in which case the filterproduct $\mathcal{M} = \prod_{i\in I} \mathcal{M}_i/D$ of the structures $\mathcal{M}_i, i \in I$, over D is defined as the factor-structure $\prod_{i\in I} \mathcal{M}_i/\sim_D$. If D is an ultrafilter on I (a filter that contain either

a or $I \smallsetminus a$ for any $a \subseteq I$) then a filterproduct over D is called an *ultraproduct*, furthermore, if all the structures $\mathcal{M}_i$ are isomorphic to some structure $\mathcal{N}$ then the ultraproduct $\prod_{i\in I} \mathcal{M}_i/D$ is called an *ultrapower* and we denote it by $\mathcal{N}^I/D$. By $\mathbf{P_f}(\mathbf{K})$ and $\mathbf{P_u}(\mathbf{K})$ we denote, correspondingly, the classes of filterproducts and ultraproducts of structures from $\mathbf{K}$.

Let $\mathbf{K}_e = \mathbf{K} \cup \{\mathcal{E}\}$, where $\mathcal{E}$ is the trivial $\mathcal{L}$-structure introduced earlier. A word of warning is needed here. Sometimes, direct products $\prod_{i\in I} \mathcal{M}_i$ are defined being equal to $\mathcal{E}$ for the empty set I (see, for example, [15]), but we elect not to do so, assuming always that I is non-empty and adding $\mathcal{E}$ to the class, if needed.

Lemma 3.6. *For any class of $\mathcal{L}$-structures $\mathbf{K}$ the following holds:*

5) $\mathbf{Qvar}(\mathbf{K}) = \mathbf{SP_f}(\mathbf{K})_{\mathbf{e}}$;

6) $\mathbf{Qvar}(\mathbf{K}) = \mathbf{SPP_u}(\mathbf{K})_{\mathbf{e}} = \mathbf{SP_uP}(\mathbf{K})_{\mathbf{e}}$;

7) $\mathbf{Qvar}(\mathbf{K}) = \mathbf{SP_uP_\omega}(\mathbf{K})_{\mathbf{e}}$;

Proof. 5) is due to Malcev [30, §11, Theorem 4]. 6) and 7) are due to Gorbunov [15, Corollary 2.3.4, Theorem 2.3.6]. □

Now we give another characterization of quasivarieties, for this we need to introduce direct limits.

Recall, that a partial ordering $(I, \leqslant)$ is *directed* if any two elements from I have an upper bound. A triple $\Lambda = (I, \mathcal{M}_i, h_{ij})$, consisting of a directed ordering $(I, \leqslant)$, a set of $\mathcal{L}$-structures $\{\mathcal{M}_i, i \in I\}$, and a set of homomorphisms $h_{ij} : \mathcal{M}_i \to \mathcal{M}_j$ $(i, j \in I,\ i \leqslant j)$, is called a *direct system* of structures $\mathcal{M}_i$, $i \in I$, if

1. h_{ii} is the identity map for every $i \in I$;
2. $h_{jk} \circ h_{ij} = h_{ik}$ for any $i, j, k \in I$ with $i \leqslant j \leqslant k$.

We call a directed system $\Lambda = (I, \mathcal{M}_i, h_{ij})$ *epimorphic* if all the homomorphisms $h_{ij} : \mathcal{M}_i \to \mathcal{M}_j$ are surjective.

Given a direct system $\Lambda = (I, \mathcal{M}_i, h_{ij})$ one can consider an equivalence relation $\equiv$ on a set $\{(m_i, i) \mid m_i \in M_i,\ i \in I\}$ defined by

$$(m_i, i) \equiv (m_j, j) \quad \Leftrightarrow \quad \exists\, k \in I,\ i, j \leqslant k, \quad h_{ik}(m_i) = h_{jk}(m_j).$$

By $\langle m, i\rangle$ we denote the equivalence class of (m, i) under $\equiv$. Now one can turn the factor-set $M = \{(m_i, i) \mid m_i \in M_i,\ i \in I\}/\equiv$ into an $\mathcal{L}$-structure $\mathcal{M}$ interpreting the constants and functions from $\mathcal{L}$ as follows:

1. if $c \in \mathcal{L}$ is a constant then $c^{\mathcal{M}} = \langle c^{\mathcal{M}_i}, i\rangle$ for an arbitrary chosen $i \in I$;
2. if $F \in \mathcal{L}$ is a function and $\langle m_1, i_1\rangle, \ldots, \langle m_{n_F}, i_{n_F}\rangle \in M$ then

   $$F^{\mathcal{M}}(\langle m_1, i_1\rangle, \ldots, \langle m_{n_F}, i_{n_F}\rangle) = F^{\mathcal{M}_j}(\langle h_{i_1 j}(m_1), i_1\rangle, \ldots, \langle h_{i_{n_F} j}(m_{n_F}), i_{n_F}\rangle)$$

   for an arbitrary chosen $j \in I$ with $i_1, \ldots, i_{n_F} \leqslant j$.

The structure $\mathcal{M}$ is well-defined, it is called the *direct limit* of the system Λ, we denote it by $\varinjlim \mathcal{M}_i$. It is easy to see that $\varinjlim \mathcal{M}_i$ has the following property. Let $i \in I$ be a fixed index. Put $J_i = \{j \in I \mid i \leqslant j\}$ a nd denote $\Lambda_i = (J_i, \mathcal{M}_j, h_{jk},\ j,k \in J_i)$. Then Λ_i is a direct system whose direct limit $\mathcal{M}^i$ is isomorphic to $\mathcal{M}$. By $\underrightarrow{\mathbf{L}}(\mathbf{K})$ and $\underrightarrow{\mathbf{L}}_s(\mathbf{K})$ we denote the class of direct and epimorphic direct limits of structures from $\mathbf{K}$.

The following result gives a characterization of quasivarieties in terms of direct limits.

Lemma 3.7. *For any class of $\mathcal{L}$-structures* $\mathbf{K}$ *the following holds:*

$$\mathbf{Qvar}(\mathbf{K}) = \mathbf{S}\underrightarrow{\mathbf{L}}_s\mathbf{P}(\mathbf{K})_{\mathbf{e}} = \underrightarrow{\mathbf{L}}_s\mathbf{SP}(\mathbf{K})_{\mathbf{e}} = \underrightarrow{\mathbf{L}}_s\mathbf{P}_{\mathbf{s}}(\mathbf{K})_{\mathbf{e}} = \underrightarrow{\mathbf{L}}\mathbf{SP}(\mathbf{K})_{\mathbf{e}}.$$

Proof. See [15, Corollary 2.3.4]. □

3.3 Universal closures

In this section we study the universal closure $\mathbf{Ucl}(\mathbf{K}) = \mathrm{Mod}(\mathrm{Th}_\forall(\mathbf{K}))$ of a given class of $\mathcal{L}$-structures $\mathbf{K}$.

Structures from $\mathbf{Ucl}(\mathbf{K})$ are determined by local properties of structures from $\mathbf{K}$. To explain precisely we need to introduce two more operators.

Recall [5, 34], that a structure $\mathcal{M}$ is *discriminated* by $\mathbf{K}$ if for any finite set W of elements from $\mathcal{M}$ there is a structure $\mathcal{N} \in \mathbf{K}$ and a homomorphism $h : \mathcal{M} \to \mathcal{N}$ whose restriction onto W is injective. Let $\mathbf{Dis}(\mathbf{K})$ be the class of $\mathcal{L}$-structures discriminated by $\mathbf{K}$. Clearly, $\mathbf{Dis}(\mathbf{K}) \subseteq \mathbf{Res}(\mathbf{K})$.

To introduce the second operator we need to describe local submodels of a structure $\mathcal{M}$. First, we replace the language $\mathcal{L}$ by a new relational language $\mathcal{L}^{rel}$, where every operational and constant symbols $F \in \mathcal{F}$ and $c \in \mathcal{C}$ are replaced, correspondingly, by a new predicate symbol R_F of arity $n_F + 1$ and a new unary predicate symbol R_c. Secondly, the structure $\mathcal{M}$ turns into a $\mathcal{L}^{rel}$-structure $\mathcal{M}^{rel}$, where the predicates $R_c^{\mathcal{M}^{rel}}$ and $R_F^{\mathcal{M}^{rel}}$ are defined by

R1) for $m \in M$ the predicate $R_c^{\mathcal{M}^{rel}}(m)$ is true in $\mathcal{M}^{rel}$ if and only if $c^{\mathcal{M}} = m$;

R2) for $m_0, m_1, \ldots, m_{n_F} \in M$ the predicate $R_F^{\mathcal{M}^{rel}}(m_0, m_1, \ldots, m_{n_F})$ is true in $\mathcal{M}^{rel}$ if and only if $F^{\mathcal{M}}(m_1, \ldots, m_{n_F}) = m_0$.

Third, if $\mathcal{L}_0$ is a finite reduct (sublanguage) of $\mathcal{L}$ then by $\mathcal{M}^{\mathcal{L}_0}$ we denote the reduct of $\mathcal{M}^{rel}$, where only predicates corresponding to constants and operations from $\mathcal{L}_0$ are survived, so $\mathcal{M}^{\mathcal{L}_0}$ is an $\mathcal{L}_0^{rel}$-structure. Now, following [30], by a *local submodel* of $\mathcal{M}$ we understand a finite substructure of $\mathcal{M}^{\mathcal{L}_0}$ for some finite reduct $\mathcal{L}_0$ of $\mathcal{L}$.

Finally, a structure $\mathcal{M}$ is *locally embeddable* into $\mathbf{K}$ if every local submodel of $\mathcal{M}$ is isomorphic to some local submodel of a structure from $\mathbf{K}$ (in the language $\mathcal{L}_0^{rel}$). By $\mathbf{L}(\mathbf{K})$ we denote the class of $\mathcal{L}$-structures locally embeddable into $\mathbf{K}$.

It is convenient for us to rephrase the notion of a local submodel in terms of formulas.

Let $\mathcal{L}'$ be a finite reduct of $\mathcal{L}$ and X a finite set of variables. A quantifier-free formula φ in $\mathcal{L}'$ is called a *diagram-formula* if φ is a conjunction of atomic formulas or their negations that satisfies the following conditions:

1) every formula $\neg(x = y)$, for each pair $(x, y) \in X^2$ with $x \neq y$, occurs in φ;

2) for each functional symbol $F \in \mathcal{L}'$ and each tuple of variables $(x_0, x_1, \ldots, x_{n_F}) \in X^{n_F+1}$ either formula $F(x_1, \ldots, x_{n_F}) = x_0$ or its negation occurs in φ;

3) for each constant symbol $c \in \mathcal{L}'$ and each $x \in X$ either $x = c$ or its negation $\neg(x = c)$ occurs in φ.

We say that φ is a *diagram-formula* in $\mathcal{L}$ if it is a diagram-formula for some finite reduct $\mathcal{L}'$ of $\mathcal{L}$ and a finite set X. The name of diagram-formulas comes from the diagrams of algebraic structures (see Section 3.4).

The following lemma is easy.

Lemma 3.8. *For any local submodel $\mathcal{N}$ of $\mathcal{M}$ there is a diagram-formula $\varphi_{\mathcal{N}}(X)$ in a finite set of variables X of cardinality $|N|$ such that $\mathcal{M} \models \varphi_{\mathcal{N}}(h(X))$ for some bijection $h : X \to N$. And conversely, if $\mathcal{M} \models \varphi(h(X))$ for some diagram-formula $\varphi(X)$ in $\mathcal{L}$ and an interpretation $h : X \to M$ then there is a local submodel $\mathcal{N}$ of $\mathcal{M}$ with the universe $h(X)$ such that $\varphi = \varphi_{\mathcal{N}}$ (up to a permutation of conjuncts).*

Corollary 3.9. An $\mathcal{L}$-structure $\mathcal{M}$ is locally embeddable into a class $\mathbf{K}$ if and only if every diagram-formula realizable in $\mathcal{M}$ is realizable also in some structure from $\mathbf{K}$.

Lemma 3.10. *For any class of $\mathcal{L}$-structures $\mathbf{K}$ the following holds:*

8) $\mathbf{Ucl}(\mathbf{K}) = \mathbf{L}(\mathbf{K})$;

9) $\mathbf{Ucl}(\mathbf{K}) = \mathbf{SP_u}(\mathbf{K})$;

10) $\mathbf{Dis}(\mathbf{K}) \subseteq \mathbf{Ucl}(\mathbf{K})$;

11) $\underset{\to}{\mathbf{L}}(\mathbf{K}) \subseteq \mathbf{Ucl}(\mathbf{K})$.

Proof. To prove 8) and 9) we show that $\mathbf{L}(\mathbf{K}) \subseteq \mathbf{SP_u}(\mathbf{K}) \subseteq \mathbf{Ucl}(\mathbf{K}) \subseteq \mathbf{L}(\mathbf{K})$. The first inclusion has been proven by Malcev [30], but we briefly discuss it for the sake of completeness. Let $\mathcal{M}$ be a structure from $\mathbf{L}(\mathbf{K})$. By Corollary 3.9 every diagram-formula φ realizable in $\mathcal{M}$ is realizable also in some structure $\mathcal{N}_\varphi$ from $\mathbf{K}$. By the Compactness Theorem the set $\Phi_{\mathcal{M}}$ of all diagram-formulas realizable in $\mathcal{M}$ is realized in some ultraproduct $\mathcal{N} = \prod_\varphi \mathcal{N}_\varphi / D$, where φ runs over $\Phi_{\mathcal{M}}$. By Lemma 3.12 the core $\mathrm{Diag}_0(\mathcal{M})$ of the diagram of $\mathcal{M}$ is also realized in $\mathcal{N}$ under an appropriate interpretation of constants $c_m, m \in M$ (see Section 3.4). Now the substructure of $\mathcal{N}$ generated by all elements $c_m, m \in M$, is isomorphic to $\mathcal{M}$. Hence $\mathcal{M} \in \mathbf{SP_u}(\mathbf{K})$.

Inclusion $\mathbf{SP_u}(\mathbf{K}) \subseteq \mathbf{Ucl}(\mathbf{K})$ follows from two known results: any universal class is closed under substructures (which is obvious) and the Los theorem

[30, 32]. To see that $\mathbf{Ucl}(\mathbf{K}) \subseteq \mathbf{L}(\mathbf{K})$ consider an arbitrary $\mathcal{M} \in \mathbf{Ucl}(\mathcal{M})$. If $\varphi(x_1, \ldots, x_n)$ is a diagram-formula which is realized in $\mathcal{M}$ then a universal sentence $\psi = \forall x_1, \ldots, x_n \neg\varphi(x_1, \ldots, x_n)$ is false in $\mathcal{M}$. Hence, there exists a structure $\mathcal{N} \in \mathbf{K}$ on which ψ is false, so $\mathcal{N} \models \neg\psi$. Therefore, $\varphi(x_1, \ldots, x_n)$ is realized in $\mathcal{N}$. By Corollary 3.9 $\mathcal{M} \in \mathbf{L}(\mathbf{K})$, as required.

To see 10) it suffices to notice that $\mathbf{Dis}(\mathbf{K}) \subseteq \mathbf{L}(\mathbf{K})$ and then apply 8).

11) follows from 9) and [15] (Theorem 1.2.9), where it is shown that $\underrightarrow{\mathbf{L}}(\mathbf{K}) \subseteq \mathbf{SP_u}(\mathbf{K})$.

□

3.4 $\mathcal{A}$-Algebras

Let $\mathcal{A}$ be a fixed $\mathcal{L}$-algebra. In this section we discuss $\mathcal{A}$-algebras — principal objects in algebraic geometry over $\mathcal{A}$. Informally, an $\mathcal{A}$-algebra is an $\mathcal{L}$-algebra with a distinguished subalgebra $\mathcal{A}$. Even though this notion seems simple, one needs to develop a formal framework to deal with $\mathcal{A}$-algebras. It will be convenient to use two equivalent approaches: one is categorical and another is logical (or axiomatic).

Definition 3.11. [Categorical] An $\mathcal{A}$-algebra is a pair $(\mathcal{B}, \lambda)$, where $\mathcal{B}$ is an $\mathcal{L}$-algebra and $\lambda : \mathcal{A} \to \mathcal{B}$ is an embedding.

For the axiomatic definition we are going to use the language of diagrams. By $\mathcal{L}_\mathcal{A}$ we denote the language $\mathcal{L} \cup \{c_a \mid a \in A\}$, which is obtained from $\mathcal{L}$ by adding a new constant c_a for every element $a \in A$.

Observe, that every $\mathcal{A}$-algebra $(\mathcal{B}, \lambda)$ can be viewed as an $\mathcal{L}_\mathcal{A}$-algebra when the constant c_a is interpreted by $\lambda(a)$.

Recall that by $\mathrm{At}_{\mathcal{L}_\mathcal{A}}(\emptyset)$ we denote the set of all atomic sentences in the language $\mathcal{L}_\mathcal{A}$. The *diagram* $\mathrm{Diag}(\mathcal{A})$ of $\mathcal{A}$ is the set of all atomic sentences from $\mathrm{At}_{\mathcal{L}_\mathcal{A}}(\emptyset)$ or their negations which are true in $\mathcal{A}$. To work with diagrams we need to define several related sets of formulas.

The *core* $\mathrm{Diag}_0(\mathcal{A})$ of the diagram $\mathrm{Diag}(\mathcal{A})$ consists of the following formulas:

- $c = c_a$ for each constant symbol $c \in \mathcal{C}$ and $a \in A$ such that $c^\mathcal{A} = a$;
- $F(c_{a_1}, \ldots, c_{a_{n_F}}) = c_{a_0}$, for each functional symbol $F \in \mathcal{F}$ and each tuple of elements $(a_0, a_1, \ldots, a_{n_F}) \in A^{n_F+1}$ such that $F^\mathcal{A}(a_1, \ldots, a_{n_F}) = a_0$;
- $c_{a_1} \neq c_{a_2}$, for each pair $(a_1, a_2) \in A^2$ such that $a_1 \neq a_2$.

The following result is easy

Lemma 3.12. *For an $\mathcal{L}$-algebra $\mathcal{A}$ the following hold:*

C1) For every $\mathcal{L}_\mathcal{A}$-structure $\mathcal{B}$ if $\mathcal{B} \models \mathrm{Diag}_0(\mathcal{A})$ then $\mathcal{B} \models \mathrm{Diag}(\mathcal{A})$;

C2) If S is a finite subset of $\mathrm{Diag}_0(\mathcal{A})$ then there is a diagram-formula $\varphi(X)$ in $\mathcal{L}$ and an interpretation $h : X \to A$ such that every formula from S occurs as a conjunct in $\varphi(h(X))$ (after replacing $h(x)$ with $c_{h(x)}$) and $\mathcal{A} \models \varphi(h(X))$;

C3) If $\varphi(X)$ is a diagram-formula in $\mathcal{L}$ and $h : X \to A$ is an interpretation such that $\mathcal{A} \models \varphi(h(X))$ then every conjunct of $\varphi(X)$ (where x is replaced with $c_{h(x)}$) belongs to $\mathrm{Diag}(\mathcal{A})$.

The following result gives an axiomatic way to describe $\mathcal{A}$-algebras.

Lemma 3.13. *Let $\mathcal{B}$ be an $\mathcal{L}$-algebra and $\lambda : A \to B$ a map. Then $(\mathcal{B}, \lambda)$ is an $\mathcal{A}$-algebra if and only if $\mathcal{B} \models \mathrm{Diag}(\mathcal{A})$, where c_a is interpreted by $\lambda(a)$ for every $a \in A$.*

Proof. Straightforward.

□

This leads to the following, equivalent, definition of $\mathcal{A}$-algebras.

Definition 3.14. [Axiomatic] An algebra $\mathcal{B}$ in the language $\mathcal{L}_\mathcal{A}$ is called an $\mathcal{A}$-algebra if $\mathcal{B} \models \mathrm{Diag}(\mathcal{A})$.

Put

$$\mathrm{Diag}^+(\mathcal{A}) = \{\varphi \in \mathrm{At}_{\mathcal{L}_\mathcal{A}}(\emptyset) \mid \mathcal{A} \models \varphi\},$$

$$\mathrm{Diag}^-(\mathcal{A}) = \mathrm{Diag}(\mathcal{A}) \smallsetminus \mathrm{Diag}^+(\mathcal{A}).$$

Let $\mathbf{Cat}(\mathcal{A})$ be the class of all $\mathcal{A}$-algebras. Since $\mathcal{A}$-algebras are $\mathcal{L}_\mathcal{A}$-structures the standard notions of a $\mathcal{L}_\mathcal{A}$-homomorphism, $\mathcal{L}_\mathcal{A}$-substructure, $\mathcal{L}_\mathcal{A}$-generating set, etc., are defined in $\mathbf{Cat}(\mathcal{A})$. Sometimes, we refer to them as to an $\mathcal{A}$-homomorphism, $\mathcal{A}$-substructure, $\mathcal{A}$-generating set, etc. Class $\mathbf{Cat}(\mathcal{A})$ with $\mathcal{A}$-homomorphism forms a category of $\mathcal{A}$-algebras.

All the operators $\mathbf{O}$ introduced in Sections 3.3 and 3.2 are defined for $\mathcal{L}_\mathcal{A}$-structures, but, a priori, the resulting $\mathcal{L}_\mathcal{A}$-algebra may not be in the class $\mathbf{Cat}(\mathcal{A})$. Nevertheless, one can check directly for each such operator $\mathbf{O}$ (with the exception of the operator $\mathbf{K} \to \mathbf{K}_e$ that adds the trivial structure $\mathcal{E}$ to $\mathbf{K}$) that $\mathbf{O}(\mathbf{Cat}(\mathcal{A})) \subseteq \mathbf{Cat}(\mathcal{A})$. Sometimes, we add the subscript $\mathcal{A}$ and write $\mathbf{O}_\mathcal{A}$ to emphasize the fact that the algebras under consideration are $\mathcal{A}$-algebras. Another, shorter, way to prove this is to show that $\mathbf{Cat}(\mathcal{A})_e$ is a quasivariety, and then these results, as well as some others, will follow for free.

Lemma 3.15. *The class $\mathbf{Cat}(\mathcal{A})_\mathbf{e}$ is a quasivariety in the language $\mathcal{L}_\mathcal{A}$ defined by the following set of quasi-identities:*

1. *$c = c_a$, for each constant symbol $c \in \mathcal{L}$ and element $a \in A$ such that $a = c^\mathcal{A}$;*

2. *$F(c_{a_1}, \ldots, c_{a_{n_F}}) = c_a$, for each functional symbol $F \in \mathcal{L}$ and each tuple $(a_1, \ldots, a_{n_F}, a) \in A^{n_F+1}$ such that $F^\mathcal{A}(a_1, \ldots, a_{n_F}) = a$;*

3. *$\forall x \, \forall y \, (c_{a_1} = c_{a_2} \to x = y)$, for each pair of elements $a_1, a_2 \in A$ with $a_1 \neq a_2$.*

Proof. It is easy to see that any $\mathcal{A}$-algebra and the trivial algebra $\mathcal{E}$ satisfy the formulas above. One needs to check the converse. Suppose $\mathcal{C}$ is an $\mathcal{L}_{\mathcal{A}}$-algebra, satisfying the formulas above. If $\mathcal{C} = \mathcal{E}$ then $\mathcal{C} \in \mathbf{Cat}(\mathcal{A})_{\mathbf{e}}$. Assume now that $\mathcal{C} \neq \mathcal{E}$. The formulas 1) and 2) show that $\mathcal{C} \models \mathrm{Diag}_0(\mathcal{A}) \cap \mathrm{Diag}^+(\mathcal{A})$, while the formulas 3) provide $\mathcal{C} \models \mathrm{Diag}_0(\mathcal{A}) \cap \mathrm{Diag}^-(\mathcal{A})$. Altogether, $\mathcal{C} \models \mathrm{Diag}_0(\mathcal{A})$, so by Lemma 3.12 $\mathcal{C} \models \mathrm{Diag}(\mathcal{A})$, as claimed. □

Corollary 3.16. Let $\mathcal{A}$ be an algebra and $\mathbf{K}$ a class of $\mathcal{A}$-algebras. Then the following holds:

1) $\mathbf{K}$ is closed under the operators $\mathbf{S}_{\mathcal{A}}$, $\mathbf{P}_{\mathcal{A}}$, $\mathbf{P}_{\omega\mathcal{A}}$, $\mathbf{P}_{\mathbf{s}\mathcal{A}}$, $\mathbf{P}_{\mathbf{f}\mathcal{A}}$, $\mathbf{P}_{\mathbf{u}\mathcal{A}}$, $\underrightarrow{\mathbf{L}}_{\mathcal{A}}$, $\underrightarrow{\mathbf{L}_s}_{\mathcal{A}}$, $\mathbf{L}_{\mathcal{A}}$;

2) every algebra in the classes $\mathbf{Pvar}_{\mathcal{A}}(\mathbf{K})$, $\mathbf{Ucl}_{\mathcal{A}}(\mathbf{K})$, $\mathbf{Res}_{\mathcal{A}}(\mathbf{K})$, and $\mathbf{Dis}_{\mathcal{A}}(\mathbf{K})$ is an $\mathcal{A}$-algebras;

3) every algebra in $\mathbf{Qvar}_{\mathcal{A}}(\mathbf{K})$, with the exception of $\mathcal{E}$, is an $\mathcal{A}$-algebra.

4 Types, Zariski topology, and coordinate algebras

In this section we introduce algebras defined by complete atomic types.

4.1 Quantifier-free types and Zariski topology

Let $\mathcal{L}$ be a functional language, T a theory in $\mathcal{L}$, and $X = \{x_1, \ldots, x_n\}$ a finite set of variables. Recall (see, for example, [32]), that a *type* in variables X of $\mathcal{L}$ over T is a consistent with T set p of formulas in $\Phi_{\mathcal{L}}(X)$, i.e, a subset $p \subseteq \Phi_{\mathcal{L}}(X)$ that can be realized in a structure from $\mathrm{Mod}(T)$.

A type p is *complete* if it is a maximal type in $\Phi_{\mathcal{L}}(X)$ with respect to inclusion. It is easy to see that if p is a maximal type in X then for every formula $\varphi \in \Phi_{\mathcal{L}}(X)$ either $\varphi \in p$ or $\neg\varphi \in p$.

Definition 4.1. A set p of atomic or negations of atomic formulas from $\Phi_{\mathcal{L}}(X)$ is called an *atomic type* in X relative to a theory T if $p \cup T$ is consistent. A maximal atomic type in $\Phi_{\mathcal{L}}(X)$ with respect to inclusion termed a *complete atomic type* of T.

It is not hard to see that if p is a complete atomic type then for every atomic formula $\varphi \in \mathrm{At}_{\mathcal{L}}(X)$ either $\varphi \in p$ or $\neg\varphi \in p$.

Example 4.2. Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $\bar{m} = (m_1, \ldots, m_n) \in M^n$. Then the set $\mathrm{atp}^{\mathcal{M}}(\bar{m})$ of atomic or negations of atomic formulas from $\Phi_{\mathcal{L}}(X)$ that are true in $\mathcal{M}$ under an interpretation $x_i \mapsto m_i$, $i = 1, \ldots, n$, is a complete atomic type relative to any theory T such that $\mathcal{M} \in \mathrm{Mod}(T)$.

We say that a complete atomic type p in variables X is realized in $\mathcal{M}$ if $p = \mathrm{atp}^{\mathcal{M}}(\bar{m})$ for some $\bar{m} \in M^n$.

Every type p in T can be realized in some model of T (i.e., a structure from $\mathrm{Mod}(T)$). If p cannot be realized in a structure $\mathcal{M}$ then we say that $\mathcal{M}$ *omits* p. There are deep results in model theory on how to construct models of T omitting a given type or a set of types.

For an atomic type $p \subseteq \Phi_{\mathcal{L}}(X)$ by p^+ and p^- we denote, correspondingly, the set of all atomic and negations of atomic formulas in p.

If S is a set of atomic formulas from $\Phi_{\mathcal{L}}(X)$ and $\mathcal{M}$ is an $\mathcal{L}$-structure then by $\mathrm{V}_{\mathcal{M}}(S)$ we denote the set $\{(m_1, \dots, m_n) \in M^n \mid \mathcal{M} \models S(m_1, \dots, m_n)\}$ of all tuples in M^n that satisfy all the formulas from S. The set $\mathrm{V}_{\mathcal{M}}(S)$ is called the *algebraic set* defined by S in $\mathcal{M}$. We refer to S as a *system of equations* in $\mathcal{L}$, and to elements of S - as *equations* in $\mathcal{L}$. Sometimes, to emphasize that formulas are from $\mathcal{L}$ we call such equations (and systems of equations) *coefficient-free equations*, meanwhile, in the case when $\mathcal{L} = \mathcal{L}_{\mathcal{A}}$, we refer to such equations as equations with coefficients in algebra $\mathcal{A}$.

Following [4] we define *Zariski topology* on $M^n, n \geq 1$, where algebraic sets form a prebasis of closed sets, i.e., closed sets in this topology are obtained from the algebraic sets by finite unions and (arbitrary) intersections.

If p is an atomic type in $\mathcal{L}$ in variables $X = \{x_1, \dots, x_n\}$ then $\mathrm{V}_{\mathcal{M}}(p^+)$ is an algebraic set in M^n. More generally, for an arbitrary type p in X by p^+ we denote the set of all positive formulas in p, i.e., all formulas in the prenex form that do not have the negation symbol.

If p is quantifier-free type, i.e., a type consisting of quantifier-free formulas, then formulas in p^+ are conjunctions and disjunctions of atomic formulas.

Lemma 4.3. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $n \in \mathbb{N}$. Then for a subset $V \subseteq M^n$ the following conditions are equivalent:*

- *V is closed in the Zariski topology on M^n;*
- *$V = \mathrm{V}_{\mathcal{M}}(p^+)$ for some quantifier-free type p in variables $\{x_1, \dots, x_n\}$.*

Here $\mathrm{V}_{\mathcal{M}}(p^+) = \{(m_1, \dots, m_n) \in M^n \mid \mathcal{M} \models p^+(m_1, \dots, m_n)\}$.

Proof. Straightforward. □

4.2 Coordinate algebras and complete types

Let $\mathcal{M}$ be an $\mathcal{L}$-algebra. For a set S of atomic formulas from $\Phi_{\mathcal{L}}(X)$ denote by $\mathrm{Rad}_{\mathcal{M}}(S)$ the set of all atomic formulas from $\Phi_{\mathcal{L}}(X)$ that hold on every tuple from $\mathrm{V}_{\mathcal{M}}(S)$. In particular, if $\mathrm{V}_{\mathcal{M}}(S) = \emptyset$ then $\mathrm{Rad}_{\mathcal{M}}(S) = \mathrm{At}_{\mathcal{L}}(X)$. It is not hard to see that $\mathrm{Rad}_{\mathcal{M}}(S)$ is a congruent set of formulas, hence it defines a congruence that we denote by $\theta_{\mathrm{Rad}(S)}$. The $\mathcal{L}$-structure $\mathcal{T}_{\mathcal{L}}(X)/\theta_{\mathrm{Rad}(S)}$ is called the *coordinate algebra* of the algebraic set $\mathrm{V}_{\mathcal{M}}(S)$. If $Y = \mathrm{V}_{\mathcal{M}}(S)$ then the coordinate algebra $\mathcal{T}_{\mathcal{L}}(X)/\theta_{\mathrm{Rad}(S)}$ is denoted by $\Gamma(Y)$ and $\mathrm{Rad}(S)$ – by $\mathrm{Rad}(Y)$.

The following result gives a characterization of the coordinate algebras over an algebra $\mathcal{M}$.

Proposition 4.4. A finitely generated $\mathcal{L}$-algebra $\mathcal{C}$ is the coordinate algebra of some non-empty algebraic set over an $\mathcal{L}$-algebra $\mathcal{M}$ if and only if $\mathcal{C}$ is separated by $\mathcal{M}$.

Proof. Let Y be an algebraic set in M^n. With a point $p = (m_1, \ldots, m_n) \in M^n$ we associate a homomorphism $h_p : \mathcal{T}_{\mathcal{L}}(X) \to \mathcal{M}$ defined by $h_p(t) = t^{\mathcal{M}}(m_1, \ldots, m_n)$. Clearly,

$$\theta_{\mathrm{Rad}(Y)} = \bigcap_{p \in Y} \ker h_p.$$

Therefore, the diagonal homomorphism $\prod_{p\in Y} : \mathcal{T}_{\mathcal{L}}(X) \to \prod_{p\in Y} \mathcal{M}$ induces a monomorphism

$$\Gamma(Y) = \mathcal{T}_{\mathcal{L}}(X)/\theta_{\mathrm{Rad}(Y)} \to \mathcal{M}^{|Y|}.$$

It follows that $\Gamma(Y) \in \mathbf{SP}(\mathcal{M})$. Now, by Lemma 3.5 $\mathbf{SP}(\mathcal{M}) = \mathbf{Res}(\mathcal{M})$, so $\Gamma(Y) \in \mathbf{Res}(\mathcal{M})$.

Suppose now that $\mathcal{C}$ is a finitely generated $\mathcal{L}$-algebra from $\mathbf{Res}(\mathcal{M})$ with a finite generating set $X = \{x_1, \ldots, x_n\}$. Let $\mathcal{C} = \langle X \mid S\rangle$ be a presentation of $\mathcal{C}$ by the generators X and relations $S \subseteq \mathrm{At}_{\mathcal{L}}(X)$. In this case $\mathcal{C}$ is isomorphic to $\mathcal{T}_{\mathcal{L}}(X)/\theta_S$. To prove that $\mathcal{C}$ is the coordinate algebra of some algebraic set over $\mathcal{M}$ it suffices to show that $\mathrm{Rad}_{\mathcal{M}}(S) = [S]$. If $(t_1 = t_2) \notin [S]$ then there exists a homomorphism $h : \mathcal{C} \to \mathcal{M}$ with $t_1^{\mathcal{M}}(h(x_1), \ldots, h(x_n)) \neq t_2^{\mathcal{M}}(h(x_1), \ldots, h(x_n))$. Obviously, $(h(x_1), \ldots, h(x_n)) \in \mathrm{V}_{\mathcal{M}}(S)$ so $(t_1 = t_2) \notin \mathrm{Rad}_{\mathcal{M}}(S)$. This shows that $\mathrm{Rad}_{\mathcal{M}}(S) = [S]$. □

Lemma 4.5. *Let p be a complete atomic type in variables X. Then:*

- *p^+ is a congruent set of formulas;*
- *$p^+ = \mathrm{Rad}_{\mathcal{M}}(p^+)$ for every $\mathcal{L}$-structure $\mathcal{M}$ with $\mathrm{V}_{\mathcal{M}}(p) \neq \emptyset$.*

Proof. Indeed, since p is realized in some model $\mathcal{M}$ of T its positive part p^+ satisfies the assumptions of Lemma 3.3, hence it is congruent. It follows that p^+ determines a congruence θ_p on $\mathcal{T}_{\mathcal{L}}(X)$. Since p is complete one has $p^+ = \mathrm{Rad}_{\mathcal{M}}(p^+)$. □

Definition 4.6. Let X be a finite set of variables and p a complete atomic type in variables X. Then the factor-algebra $\mathcal{T}_{\mathcal{L}}(X)/\theta_p$ of the free $\mathcal{L}$-algebra $\mathcal{T}_{\mathcal{L}}(X)$ is termed the algebra defined by the type p and the tuple $(x_1/\theta_p, \ldots, x_n/\theta_p)$ is called a generic point of p.

Clearly, any complete atomic type p in variables X in a theory T is realized in the factor-algebra $\mathcal{T}_{\mathcal{L}}(X)/\theta_p$ at the generic point $\bar{x} = (x_1/\theta_p, \ldots, x_n/\theta_p)$, so

$$\mathrm{atp}^{\mathcal{T}_{\mathcal{L}}(X)/\theta_p}(\bar{x}) \; = \; p.$$

Indeed, for any atomic formula $t_1 = t_2$, where $t_1, t_2 \in \mathrm{T}_{\mathcal{L}}(X)$ one has $(t_1 = t_2) \in p$ if and only if $t_1 \sim_{\theta_p} t_2$, which is equivalent to the condition $\mathcal{T}_{\mathcal{L}}(X)/\theta_p \models (t_1 =$

t_2) under the interpretation $x_i \mapsto x_i/\theta_p$. The generic point $(x_1/\theta_p, \ldots, x_n/\theta_p)$ satisfies the following universal property. If p is realized in some $\mathcal{L}$-structure $\mathcal{M}$ at $(m_1, \ldots, m_n) \in M^n$ then the map $x_1 \to m_1, \ldots, x_n \to m_n$ extends to a homomorphism $\mathcal{T}_{\mathcal{L}}(X)/\theta_p \to \mathcal{M}$.

Lemma 4.7. *Let T be a universally axiomatized theory in $\mathcal{L}$. Then for any finitely generated $\mathcal{L}$-structure $\mathcal{M}$ the following conditions are equivalent:*

1) *$\mathcal{M} \in \mathrm{Mod}(T)$;*

2) *$\mathcal{M} = \mathcal{T}_{\mathcal{L}}(X)/\theta_p$ for some complete atomic type p in T.*

Proof. Let $X = \{x_1, \ldots, x_n\}$ be a finite set and $\langle X \mid S \rangle$ a presentation of an $\mathcal{L}$-structure $\mathcal{M}$, i.e., $\mathcal{M} \cong \mathcal{T}_{\mathcal{L}}(X)/\theta_S$. If $p = \mathrm{atp}^{\mathcal{M}}(\bar{x})$, $\bar{x} = (x_1, \ldots, x_n)$ then $[S] = p^+$ and $\mathcal{T}_{\mathcal{L}}(X)/\theta_p \cong \mathcal{T}_{\mathcal{L}}(X)/\theta_S \cong \mathcal{M}$. Therefore, 1) implies 2).

To prove the converse, let p be an atomic type in T. We need to show that $\mathcal{T}_{\mathcal{L}}(X)/\theta_p \in \mathrm{Mod}(T)$. Since p is a type in T there exists a model $\mathcal{N} \in \mathrm{Mod}(T)$ and a tuple of elements $\bar{y} = (y_1, \ldots, y_n) \in N^n$ such that $p = \mathrm{atp}^{\mathcal{N}}(\bar{y})$. If $\mathcal{N}'$ is a substructure of $\mathcal{N}$ generated by $y_1, \ldots, y_n$ then $\mathcal{T}_{\mathcal{L}}(X)/\theta_p \cong \mathcal{N}'$. Since the theory T is axiomatized by a set of universal sentences one has $\mathcal{N}' \in \mathrm{Mod}(T)$. Hence, $\mathcal{T}_{\mathcal{L}}(X)/\theta_p \in \mathrm{Mod}(T)$. □

4.3 Equationally Noetherian algebras

The notion of equationally Noetherian groups was introduced in [4] and [7].

Let $\mathcal{B}$ be an algebra. For every natural number n we consider Zariski topology on B^n.

A subset $Y \subseteq B^n$ is called *reducible* if it is a union of two proper closed subsets, otherwise, it is called *irreducible*.

It is not hard to see that an algebraic set $Y \subseteq B^n$ is irreducible if and only if it is not a finite union of proper algebraic subsets.

Recall, that a topological space is called *Noetherian* if it satisfies the descending chain condition on closed subsets.

Remark 4.8. Let $(W, \mathfrak{T})$ be a topological space, $\mathfrak{A}$ a prebase of closed subsets of $\mathfrak{T}$, and $\mathfrak{B}$ the base of $\mathcal{T}$, formed by the finite unions of sets from $\mathfrak{A}$. Suppose that $\mathfrak{A}$ is closed under finite intersections. Then the following conditions are equivalent:

- the topological space $(W, \mathfrak{T})$ is Noetherian;
- $\mathfrak{A}$ satisfies the descending chain condition.

In this case

1) the base $\mathfrak{B}$ contains all closed sets in the topology $\mathfrak{T}$;

2) any closed set Y in $\mathfrak{T}$ is a finite union of irreducible closed sets from $\mathfrak{A}$ (*irreducible components*): $Y = Y_1 \cup \ldots \cup Y_m$. Moreover, if $Y_i \not\subseteq Y_j$ for $i \neq j$ then this decomposition is unique up to a permutation of components.

Definition 4.9 (No coefficients)**.** An algebra $\mathcal{B}$ is *equationally Noetherian*, if for any natural number n and any system of equations $S \subseteq \mathrm{At}_{\mathcal{L}}(x_1, \ldots, x_n)$ there exists a finite subsystem $S_0 \subseteq S$ such that $\mathrm{V}_{\mathcal{B}}(S) = \mathrm{V}_{\mathcal{B}}(S_0)$.

Definition 4.10 (Coefficients in $\mathcal{A}$)**.** An $\mathcal{A}$-algebra $\mathcal{B}$ is $\mathcal{A}$-*equationally Noetherian* if for any natural number n and any system of equations $S \subseteq \mathrm{At}_{\mathcal{L}_{\mathcal{A}}}(x_1, \ldots, x_n)$ there exists a finite subsystem $S_0 \subseteq S$ such that $\mathrm{V}_{\mathcal{B}}(S) = \mathrm{V}_{\mathcal{B}}(S_0)$.

Lemma 4.11. *An ($\mathcal{A}$-) algebra $\mathcal{B}$ is ($\mathcal{A}$-) equationally Noetherian if and only if for any natural number n Zariski topology on B^n is Noetherian.*

Proof. We prove the lemma for coefficient-free equations, a similar argument gives the result for equations with coefficients in $\mathcal{A}$.

Assume $\mathcal{B}$ is equationally Noetherian and consider a descending chain of closed subsets $Y_1 \supseteq Y_2 \supseteq Y_3 \supseteq \ldots$ of algebraic sets in $\mathcal{B}^n$. Taking the radicals one gets an ascending chain of subalgebras $\mathrm{Rad}(Y_1) \subseteq \mathrm{Rad}(Y_2) \subseteq \mathrm{Rad}(Y_3) \subseteq \ldots$. Put $S = \bigcup_i \mathrm{Rad}(Y_i)$. By our assumption the system S is equivalent to some finite subsystem $S_0 \subseteq S$. Clearly, $S_0 \subseteq \mathrm{Rad}(Y_i)$ for some index i. Therefore, the chains before stabilize.

Suppose now that for any natural number n Zariski topology on B^n is Noetherian. Let $S \subseteq \mathrm{At}_{\mathcal{L}}(x_1, \ldots, x_n)$ be an arbitrary system of equations in variables $\{x_1, \ldots, x_n\}$. Let $(t_1 = s_1) \in S$. If $\mathrm{V}_{\mathcal{B}}(S) = \mathrm{V}_{\mathcal{B}}(\{t_1 = s_1\})$ then there is nothing to prove. Otherwise, there is an atomic formula $(t_2 = s_2) \in S \backslash \{t_1 = s_1\}$ with $\mathrm{V}(\{t_1 = s_1\}) \supsetneq \mathrm{V}(\{t_1 = s_1, t_2 = s_2\})$. Repeating this process one can produce a descending chain of closed subsets in B^n. Since B^n is Noetherian the chain is finite, so $\mathrm{V}_{\mathcal{B}}(S) = \mathrm{V}_{\mathcal{B}}(S_0)$ for some finite subsystem S_0 of S. □

The following result follows immediately from Lemma 4.11 and Remark 4.8.

Theorem 4.12. *Let $\mathcal{B}$ be an ($\mathcal{A}$-)equationally Noetherian ($\mathcal{A}$-)algebra. Then any algebraic set $Y \subseteq B^n$ is a finite union of irreducible algebraic sets (irreducible components): $Y = Y_1 \cup \ldots \cup Y_m$. Moreover, if $Y_i \not\subseteq Y_j$ for $i \neq j$ then this decomposition is unique up to a permutation of components.*

Now we give a characterization of the coordinate algebras of irreducible algebraic sets over an arbitrary algebra $\mathcal{B}$.

Lemma 4.13. *Let Y be an irreducible algebraic set over $\mathcal{B}$. Then the coordinate algebra $\Gamma(Y)$ is discriminated by $\mathcal{B}$.*

Proof. Indeed, let $Y = \mathrm{V}(S)$ and $\Gamma(Y) = \mathcal{T}_{\mathcal{L}}(X)/\theta_{\mathrm{Rad}(Y)}$. Suppose, to the contrary, that there exist such atomic formulas $(t_i = s_i) \in \mathrm{At}_{\mathcal{L}}(X)$, $(t_i = s_i) \notin \mathrm{Rad}(Y)$, $i = 1, \ldots, m$, such that for any homomorphism $h : \Gamma(Y) \to \mathcal{B}$ there exists an index $i \in \{1, \ldots, m\}$ for which $h(t_i/\theta_{\mathrm{Rad}(S)}) = h(s_i/\theta_{\mathrm{Rad}(S)})$. This implies that for any $p \in Y$ there exists an index $i \in \{1, \ldots, m\}$ with $t_i^{\mathcal{B}}(p) = s_i^{\mathcal{B}}(p)$. Put $Y_i = \mathrm{V}(S \cup \{t_i = s_i\})$, $i = 1, \ldots, m$. Then $Y = Y_1 \cup \ldots \cup Y_m$ and the sets $Y_1, \ldots, Y_m$ are proper closed subsets of Y — contradiction with irreducibility of Y. This shows that $\Gamma(Y)$ is discriminated by $\mathcal{B}$. □

The converse of this result also holds.

Lemma 4.14. *Let $\mathcal{C}$ be a finitely generated $\mathcal{L}$-algebra. If $\mathcal{C}$ is discriminated by an $\mathcal{L}$-algebra $\mathcal{B}$ then $\mathcal{C}$ is the coordinate algebra of some algebraic set over $\mathcal{B}$.*

Proof. Since $\mathbf{Dis}(\mathcal{B}) \subseteq \mathbf{Res}(\mathcal{B})$ then by Proposition 4.4 $\mathcal{C} = \Gamma(Y)$ for some algebraic set Y over $\mathcal{B}$. To prove the result it suffices to reverse the argument in Lemma 4.13. Indeed, suppose $Y = Y_1 \cup \ldots \cup Y_m$ for some proper algebraic subsets Y_i. From $Y_i \subset Y$ and $Y_i \neq Y$ follows that $\mathrm{Rad}(Y) \subset \mathrm{Rad}(Y_i)$ and $\mathrm{Rad}(Y) \neq \mathrm{Rad}(Y_i)$, so there exists an atomic formula $(t_i = s_i) \in \mathrm{Rad}(Y_i) \backslash \mathrm{Rad}(Y)$, $i = 1, \ldots, m$. This implies that there is no any homomorphism $h : \Gamma(Y) \to \mathcal{B}$ with $h(t_i/\theta_{\mathrm{Rad}(Y)}) \neq h(s_i/\theta_{\mathrm{Rad}(Y)})$ for all $i = 1, \ldots, m$, — contradiction with $\mathcal{C} \in \mathbf{Dis}(B)$. ☐

Theorem 4.15. *Let $\mathcal{B}$ be an $\mathcal{L}$-algebra and $\mathcal{C}$ a finitely generated $\mathcal{L}$-algebra. Then $\mathcal{C}$ is the coordinate algebra of some irreducible algebraic set over $\mathcal{B}$ if and only if $\mathcal{C}$ is discriminated by $\mathcal{B}$.*

Proof. Follows immediately from Lemmas 4.13 and 4.14, and Remark 4.8. ☐

A similar argument gives the result for $\mathcal{A}$-algebras.

Theorem 4.16. *Let $\mathcal{B}$ be an $\mathcal{A}$-algebra and $\mathcal{C}$ a finitely generated $\mathcal{A}$-algebra. Then $\mathcal{C}$ is the coordinate algebra of some irreducible algebraic set over $\mathcal{B}$ if and only if $\mathcal{C}$ is $\mathcal{A}$-discriminated by $\mathcal{B}$.*

5 Limit algebras

5.1 Direct systems of formulas and limit algebras

In this section we discuss limit $\mathcal{L}$-algebras. We need the following notation. For a formula $\varphi \in \Phi_{\mathcal{L}}(X)$ and a map $\gamma : X \to X'$ from X into a set of variables X' by $\varphi(\gamma(X))$ we denote the formula obtained from φ by the substitution $x \to \gamma(x)$ for every $x \in X$.

Definition 5.1. A triple $\Lambda = (I, \varphi_i, \gamma_{ij})$ is called a *direct system of formulas* in $\mathcal{L}$ if

1. $\langle I, \leqslant \rangle$ is a directed ordering;
2. for each $i \in I$ there is a finite reduct $\mathcal{L}_i$ of $\mathcal{L}$ and a finite set of variables X_i such that φ_i is a consistent diagram-formula in $\mathcal{L}_i$ in variables X_i;
3. $\gamma_{ij} : X_i \to X_j$ is a map defined for every pair of indices $i, j \in I$, $i \leqslant j$, such that:
   - γ_{ii} is the identical map for every $i \in I$;
   - $\gamma_{jk} \circ \gamma_{ij} = \gamma_{ik}$ for every $i, j, k \in I$, $i \leqslant j \leqslant k$;
   - all conjuncts of $\varphi_i(\gamma_{ij}(X_i))$ are also conjuncts of $\varphi_j(X_j)$;

4. for any $c \in \mathcal{L}$ there exists $i \in I$ such that φ_i contains a conjunct of the type $x_i = c$, where $x_i \in X_i$;

5. for any functional symbol $F \in \mathcal{L}$, any $i \in I$, and any tuple of variables $(x_1, \ldots, x_{n_F}) \in X_i^{n_F}$ there is $j \in I$, $i \leqslant j$ such that φ_j contains a conjunct of the type $F(\gamma_{ij}(x_1), \ldots, \gamma_{ij}(x_{n_F})) = x_j$, where $x_j \in X_j$.

Let $\Lambda = (I, \varphi_i, \gamma_{ij})$ be a direct system of formulas in $\mathcal{L}$. Define a factor-set $L(\Lambda) = \{(x_i, i),\ x_i \in X_i,\ i \in I\}/\equiv$, where

$$(x_i, i) \equiv (x_j, j) \quad \Leftrightarrow \quad \exists\, k \in I,\ i, j \leqslant k, \quad \gamma_{ik}(x_i) = \gamma_{jk}(x_j).$$

By $\langle x, i\rangle$ we denote the equivalence class of an element (x, i), $x \in X_i$, $i \in I$, relative to $\equiv$.

We turn the set $L(\Lambda)$ into an $\mathcal{L}$-algebra interpreting constants and operations from $\mathcal{L}$ on $L(\Lambda)$ as follows:

1. if $c \in \mathcal{L}$ is a constant symbol then $c^{L(\Lambda)} = \langle x_i, i\rangle$, where $i \in I$ is an arbitrary index such that the conjunction φ_i contains an atomic formula of the type $x_i = c$, with $x_i \in X_i$;

2. if $F \in \mathcal{L}$ is a symbol of operation and $\langle x_1, i_1\rangle, \ldots, \langle x_{n_F}, i_{n_F}\rangle \in L(\Lambda)$ then $F^{L(\Lambda)}(\langle x_1, i_1\rangle, \ldots, \langle x_{n_F}, i_{n_F}\rangle) = \langle x_j, j\rangle$, where $j \in I$, $i_1, \ldots, i_{n_F} \leqslant j$ and such that the conjunction φ_j contains a conjunct $F(\gamma_{i_1 j}(x_1), \ldots, \gamma_{i_{n_F} j}(x_{n_F})) = x_j$, $x_j \in X_j$.

Lemma 5.2. *The constants $c^{L(\Lambda)}$ and operations $F^{L(\Lambda)}$ are well-defined.*

Proof. Let $c \in \mathcal{L}$ be a constant symbol from $\mathcal{L}$. Then, from the definition of the direct system of formulas, there exists $i \in I$ such that φ_i contains a conjunct $x_i = c$ for some $x_i \in X_i$. Suppose that there exists another index $j \in I$ for which φ_j contains a conjunct $x_j = c$, $x_j \in X_j$. Since $\leqslant$ is a direct order on I then there exists $k \in I$ such that $i, j \leqslant k$. The formula φ_k contains conjuncts $\gamma_{ik}(x_i) = c$ and $\gamma_{jk}(x_j) = c$. Since φ_k is realizable one has $\gamma_{ik}(x_i) = \gamma_{jk}(x_j)$, so $(x_i, i) \equiv (x_j, j)$. This shows that $c^{L(\Lambda)}$ is well-defined.

Let $F \in \mathcal{L}$ be a functional symbol and $\langle x_1, i_1\rangle, \ldots, \langle x_{n_F}, i_{n_F}\rangle \in L(\Lambda)$. There exists $j_0 \in I$ such that $i_1, \ldots, i_{n_F} \leqslant j_0$, in particular, $\gamma_{i_1 j_0}(x_1), \ldots, \gamma_{i_{n_F} j_0}(x_{n_F}) \in X_{j_0}$. Then by the definition of the direct system there exists $j \in I$ such that $j_0 \leqslant j$ and φ_j contains a conjunct $F(\gamma_{j_0 j}(\gamma_{i_1 j_0}(x_1)), \ldots, \gamma_{j_0 j}(\gamma_{i_{n_F} j_0}(x_{n_F})) = x_j$, $x_j \in X_j$. Since $\gamma_{j_0 j}(\gamma_{i_k j_0}(x_k)) = \gamma_{i_k j}(x_k)$, $k = 1, \ldots, n_F$, and $i_1, \ldots, i_{n_F} \leqslant j$ then $F^{L(\Lambda)}$ is defined on $\langle x_1, i_1\rangle, \ldots, \langle x_{n_F}, i_{n_F}\rangle$.

Suppose there exists another $i \in I$ such that $i_1, \ldots, i_{n_F} \leqslant i$ and φ_i contains a conjunct $F(\gamma_{i_1 i}(x_1), \ldots, \gamma_{i_{n_F} i}(x_{n_F})) = x_i$ for some $x_i \in X_i$. Then there exists $k \in I$ such that $i, j \leqslant k$ and φ_k contains the conjuncts $F(\gamma_{i_1 k}(x_1), \ldots, \gamma_{i_{n_F} k}(x_{n_F})) = \gamma_{jk}(x_j)$ and $F(\gamma_{i_1 k}(x_1), \ldots, \gamma_{i_{n_F} k}(x_{n_F})) = \gamma_{ik}(x_i)$. The diagram-formula φ_k is consistent, hence $\gamma_{ik}(x_i) = \gamma_{jk}(x_j)$ (otherwise φ_k should contain $\gamma_{ik}(x_i) \neq \gamma_{jk}(x_j)$ which is impossible), therefore $(x_i, i) \equiv (x_j, j)$.

It is left to show that the value $F^{L(\Lambda)}(\langle x_1, i_1\rangle, \ldots, \langle x_{n_F}, i_{n_F}\rangle)$ does not depend on the representatives (x_k, i_k) in the equivalence classes $\langle x_k, i_k\rangle$, $k = 1, \ldots, n_F$. The argument is similar to the one above and we omit it. $\square$

Definition 5.3. Let $\Lambda = (I, \varphi_i, \gamma_{ij})$ be a direct system of formulas in $\mathcal{L}$. Then the set $L(\Lambda)$ with the constants $c^{L(\Lambda)}$ and operations $F^{L(\Lambda)}$ defined above for $c, F \in \mathcal{L}$ is an $\mathcal{L}$-structure termed the limit algebra of Λ or a limit algebra in $\mathcal{L}$.

Lemma 5.4. *Let $\Lambda = (I, \varphi_i, \gamma_{ij})$ be a direct system of formulas in $\mathcal{L}$. Then all formulas φ_i, $i \in I$, hold in the limit algebra $L(\Lambda)$ under the interpretation $x \mapsto \langle x, i\rangle$, $x \in X_i$, $i \in I$.*

Proof. The result follows directly from the construction of the limit algebra $L(\Lambda)$. $\square$

Lemma 5.5. *Let $\Lambda = (I, \varphi_i, \gamma_{ij})$ be a direct system of formulas in $\mathcal{L}$. Suppose $\{\mathcal{B}_i, i \in I\}$ is a family of $\mathcal{L}$-algebras such that the formula φ_i can be realized in $\mathcal{B}_i$, $i \in I$. Then there is an ultrafilter D over I such that the algebra $L(\Lambda)$ embeds into the ultraproduct $\prod_{i\in I} \mathcal{B}_i/D$.*

Proof. By the conditions of the lemma for every $i \in I$ the formula φ_i holds in $\mathcal{B}_i$ under some interpretation $h_i : X_i \to B_i$. Define a map

$$f_0 : \{(x, i), x \in X_i, i \in I\} \to \prod_{i\in I} \mathcal{B}_i$$

such that $f_0(x, i) = b \in \prod_{i\in I} \mathcal{B}_i$, where $b(j) = h_j(\gamma_{ij}(x))$, if $i \leqslant j$, and $b(j) = h_i(x)$, if $i > j$. To define an ultrafilter D over I put $J_i = \{j \in I, i \leqslant j\}$, $i \in I$. Since $\langle I, \leqslant\rangle$ is a direct ordering any finite intersection of sets from $D_0 = \{J_i \mid i \in I\}$ contains a set from D_0. Therefore, there is an ultrafilter D on I such that $D_0 \subseteq D$. Now, we define a map $f : L(\Lambda) \to \prod_{i\in I} \mathcal{B}_i/D$ by the rule: $f(\langle x, i\rangle) = f_0(x, i)/D$, $\langle x, i\rangle \in L(\Lambda)$.

It is not hard to verify that the map f is well-defined and is an injective $\mathcal{L}$-homomorphism. $\square$

Definition 5.6. Let $\Lambda = (I, \varphi_i, \gamma_{ij})$ be a direct system of formulas in $\mathcal{L}$ and $\mathcal{B}$ and $\mathcal{L}$-algebra. If every φ_i can be realized in $\mathcal{B}$ then the limit algebra $L(\Lambda)$ is called a limit algebra over $\mathcal{B}$. In this case we denote $L(\Lambda)$ by $\mathcal{B}_\Lambda$.

Corollary 5.7. Let $\Lambda = (I, \varphi_i, \gamma_{ij})$ be a direct system of formulas in $\mathcal{L}$, $\mathcal{B}$ an $\mathcal{L}$-algebra, and $\mathcal{B}_\Lambda$ a limit algebra over $\mathcal{B}$. Then there exists an ultrafilter D over I such that the limit algebra $\mathcal{B}_\Lambda$ embeds into the ultrapower $\mathcal{B}^I/D$ of $\mathcal{B}$.

The next result explain why limit algebras over $\mathcal{B}$ have this name.

Lemma 5.8. *Let $\mathcal{C}$ be a limit algebra over $\mathcal{B}$. Then $\mathcal{C}$ (viewed as a structure in the relational language $\mathcal{L}^{rel}$) is the limit of a direct system of local submodels of $\mathcal{B}$.*

Proof. Straightforward. $\square$

Let $X = \{x_b, b \in B\}$ be a set of variables indexed by elements from B. Now let I be the set of all pairs $(\mathcal{L}', X')$, where $\mathcal{L}'$ is a finite reduct of the language $\mathcal{L}$ and X' a finite subset of X. Denote by $\mathcal{B}'$ $\mathcal{L}'$-reduct of $\mathcal{B}$ and by $\varphi_{(\mathcal{L}',X')}$ the conjunction of all formulas ϕ such that (i) ϕ or $\neg\phi$ is in the core diagram $\mathrm{Diag}_0(\mathcal{B}')$, (ii) $V(\phi) \subseteq X'$, (iii) $\mathcal{B} \models \phi$ under the interpretation $x_b \mapsto b$, $x_b \in X'$. Clearly, $\varphi_{(\mathcal{L}',X')}$ is a diagram-formula. Conversely, every diagram-formula realizable in $\mathcal{B}$ can be obtained in the form of $\varphi_{(\mathcal{L}',X')}$. Define $(\mathcal{L}', X') \leqslant (\mathcal{L}'', X'')$ if and only if when $\mathcal{L}' \subseteq \mathcal{L}''$ and $X' \subseteq X''$.

It is easy to see that $(I, \leqslant)$ is a direct ordering. Define the maps $\gamma_{(\mathcal{L}',X'),(\mathcal{L}'',X'')}$, $(\mathcal{L}', X') \leqslant (\mathcal{L}'', X'')$, as the identical maps, i.e., $\gamma_{(\mathcal{L}',X'),(\mathcal{L}'',X'')}(x_b) = x_b$ for all $x_b \in X'$. Straightforward verification shows that

$$\Lambda^{\mathcal{B}} = (\{(\mathcal{L}', X')\}, \varphi_{(\mathcal{L}',X')}, \gamma_{(\mathcal{L}',X'),(\mathcal{L}'',X'')})$$

is a direct system of formulas in $\mathcal{L}$.

Lemma 5.9. *Let $\mathcal{B}$ be an $\mathcal{L}$-algebra and $\Lambda^{\mathcal{B}}$ the direct system defined above. Then $L(\Lambda^{\mathcal{B}}) \cong \mathcal{B}$.*

Proof. Notice that for every $b_1, b_2 \in B$ and $i, j \in I$ the equality $(x_{b_1}, i) \equiv (x_{b_2}, j)$ holds in the limit algebra $L(\Lambda^{\mathcal{B}})$ if and only if $b_1 = b_2$. Therefore, the map $f : \mathcal{B} \to L(\Lambda^{\mathcal{B}})$, defined by $f(b) = (x_b, i)$ for any $i \in I$, is a bijection. It is easy to check that f is an $\mathcal{L}$-homomorphism. □

Lemma 5.10. *Let $\mathcal{B}$ and $\mathcal{C}$ be $\mathcal{L}$-algebras. If $\mathrm{Th}_\exists(\mathcal{B}) \supseteq \mathrm{Th}_\exists(\mathcal{C})$ then $\mathcal{C}$ is isomorphic to some limit algebra over $\mathcal{B}$.*

Proof. By Lemma 5.9 $L(\Lambda^{\mathcal{C}}) \cong \mathcal{C}$. The inclusion $\mathrm{Th}_\exists(\mathcal{B}) \supseteq \mathrm{Th}_\exists(\mathcal{C})$ shows that all diagram-formulas in the direct system $\Lambda^{\mathcal{C}}$ are realizable in $\mathcal{B}$. Hence $L(\Lambda^{\mathcal{C}})$ is a limit algebra over $\mathcal{B}$. □

5.2 Limit $\mathcal{A}$-algebras

In this section we discuss limit algebras in the category of $\mathcal{A}$-algebras.

Definition 5.11. Let $\mathcal{A}$ be an $\mathcal{L}$-algebra and $\mathcal{L}_{\mathcal{A}}$ the language $\mathcal{L}$ with constants from $\mathcal{A}$. If $\mathcal{B}$ is an $\mathcal{A}$-algebra and Λ a direct system of formulas in $\mathcal{L}_{\mathcal{A}}$ then the algebra $\mathcal{B}_\Lambda$ is called a limit $\mathcal{A}$-algebra over $\mathcal{B}$.

Lemma 5.12. *Let $\mathcal{A}$ be an $\mathcal{L}$-algebra, $\mathcal{B}$ an $\mathcal{A}$-algebra, and Λ a direct system of formulas in $\mathcal{L}_A$. Then the limit algebra $\mathcal{B}_\Lambda$ is an $\mathcal{A}$-algebra, i.e., $\mathcal{B}_\Lambda \models \mathrm{Diag}(\mathcal{A})$.*

Proof. It is not hard to prove the result directly from definitions. However, it follows immediately from Corollary 5.7.

□

Since, in the notation above, the limit algebra $\mathcal{B}_\Lambda$ is an $\mathcal{A}$-algebra, all the results from Section 5.1 hold (after an obvious adjustment) in the category of $\mathcal{A}$-algebras. We just mention these results without proofs.

Corollary 5.13. Let $\mathcal{B}$ be an $\mathcal{A}$-algebra in the language $\mathcal{L}_\mathcal{A}$ and $\mathcal{B}_\Lambda$ the limit $\mathcal{A}$-algebra over $\mathcal{B}$ relative to the direct system $\Lambda = (I, \varphi_i, \gamma_{ij})$. Then there exists an ultrafilter D over I such that $\mathcal{B}_\Lambda$ $\mathcal{A}$-embeds into the ultrapower $\mathcal{B}^I/D$ of the algebra $\mathcal{B}$.

Corollary 5.14. Let $\mathcal{B}$ be an $\mathcal{A}$-algebra in $\mathcal{L}_\mathcal{A}$ and $\Lambda_\mathcal{A}^\mathcal{B}$ be a direct system of formulas in $\mathcal{L}_A$, corresponding to $\mathcal{B}$ (see Lemma 5.9). Then $L(\Lambda_\mathcal{A}^\mathcal{B}) \cong_\mathcal{A} \mathcal{B}$.

Corollary 5.15. Let $\mathcal{A}$ be an $\mathcal{L}$-algebra, $\mathcal{B}$ and $\mathcal{C}$ $\mathcal{A}$-algebras and $\mathrm{Th}_{\exists,\mathcal{A}}(\mathcal{B}) \supseteq \mathrm{Th}_{\exists,\mathcal{A}}(\mathcal{C})$. Then $\mathcal{C}$ is $\mathcal{A}$-isomorphic to some $\mathcal{A}$-algebra which is a limit algebra over $\mathcal{B}$.

6 Unification Theorems

Theorem A *[No coefficients] Let $\mathcal{B}$ be an equationally Noetherian algebra in a functional language $\mathcal{L}$. Then for a finitely generated algebra $\mathcal{C}$ of $\mathcal{L}$ the following conditions are equivalent:*

1. $\mathrm{Th}_\forall(\mathcal{B}) \subseteq \mathrm{Th}_\forall(\mathcal{C})$, *i.e.*, $\mathcal{C} \in \mathbf{Ucl}(\mathcal{B})$;
2. $\mathrm{Th}_\exists(\mathcal{B}) \supseteq \mathrm{Th}_\exists(\mathcal{C})$;
3. *$\mathcal{C}$ embeds into an ultrapower of $\mathcal{B}$;*
4. *$\mathcal{C}$ is discriminated by $\mathcal{B}$;*
5. *$\mathcal{C}$ is a limit algebra over $\mathcal{B}$;*
6. *$\mathcal{C}$ is defined by a complete atomic type in the theory* $\mathrm{Th}_\forall(\mathcal{B})$ *in $\mathcal{L}$;*
7. *$\mathcal{C}$ is the coordinate algebra of an irreducible algebraic set over $\mathcal{B}$ defined by a system of coefficient-free equations.*

Theorem B *[With coefficients] Let $\mathcal{A}$ be an algebra in a functional language $\mathcal{L}$ and $\mathcal{B}$ an $\mathcal{A}$-equationally Noetherian $\mathcal{A}$-algebra. Then for a finitely generated $\mathcal{A}$-algebra $\mathcal{C}$ the following conditions are equivalent:*

1. $\mathrm{Th}_{\forall,\mathcal{A}}(\mathcal{B}) \subseteq \mathrm{Th}_{\forall,\mathcal{A}}(\mathcal{C})$, *i.e.*, $\mathcal{C} \in \mathbf{Ucl}_\mathcal{A}(\mathcal{B})$;
2. $\mathrm{Th}_{\exists,\mathcal{A}}(\mathcal{B}) \supseteq \mathrm{Th}_{\exists,\mathcal{A}}(\mathcal{C})$;
3. *$\mathcal{C}$ $\mathcal{A}$-embeds into an ultrapower of $\mathcal{B}$;*
4. *$\mathcal{C}$ is $\mathcal{A}$-discriminated by $\mathcal{B}$;*
5. *$\mathcal{C}$ is a limit algebra over $\mathcal{B}$;*
6. *$\mathcal{C}$ is an algebra defined by a complete atomic type in the theory* $\mathrm{Th}_{\forall,\mathcal{A}}(\mathcal{B})$ *in the language $\mathcal{L}_\mathcal{A}$;*

7. *$\mathcal{C}$ is the coordinate algebra of an irreducible algebraic set over $\mathcal{B}$ defined by a system of equations with coefficients in $\mathcal{A}$.*

Proof. We prove here only Theorem A, the argument for Theorem B is similar and we omit it. Equivalence 1) $\Longleftrightarrow$ 2) is the standard result in mathematical logic.

Equivalence 1) $\Longleftrightarrow$ 3) has been proven in Lemma 3.10 (in the form $\mathbf{Ucl}(\mathcal{B}) = \mathbf{SP_u}(\mathcal{B})$).

Equivalence 1) $\Longleftrightarrow$ 6) has been proven in Lemma 4.7.

To see that 1) is equivalent to 5) observe first that by Corollary 5.7 one has 5) $\Longrightarrow$ 3), hence 5) $\Longrightarrow$ 1). The converse implication 1) $\Longrightarrow$ 5) follows from Lemma 5.10.

Implication 4) $\Longrightarrow$ 1) follows from $\mathbf{Dis}(\mathcal{B}) \subseteq \mathbf{Ucl}(\mathcal{B})$ (see Lemma 3.10).

Now we prove the converse implication 1) $\Longrightarrow$ 4). Suppose that $\mathcal{C} \notin \mathbf{Dis}(\mathcal{B})$. It suffices to show that $\mathcal{C} \notin \mathbf{Ucl}(\mathcal{B})$. Let $X = \{x_1, \ldots, x_n\}$ be a finite set of generators of $\mathcal{C}$ and $\langle X \mid S \rangle$ a presentation of $\mathcal{C}$ in the generators X, where $S \subseteq \mathrm{At}_{\mathcal{L}}(X)$. The latter means that $\mathcal{C} \simeq \mathcal{T}_{\mathcal{L}}(X)/\theta_S$.

Since $\mathcal{B}$ does not discriminate $\mathcal{C}$ there are atomic formulas $(t_i = s_i) \in \mathrm{At}_{\mathcal{L}}(X)$, $(t_i = s_i) \notin [S]$, $i = 1, \ldots, m$, such that for any homomorphism $h : \mathcal{C} \to \mathcal{B}$ there is an index $i \in \{1, \ldots, m\}$ for which $h(t_i/\theta_S) = h(s_i/\theta_S)$. This means that for any point $p \in \mathrm{V}_{\mathcal{B}}(S)$ there is an index $i \in \{1, \ldots, m\}$, with $t_i^{\mathcal{B}}(p) = s_i^{\mathcal{B}}(p)$. Since $\mathcal{B}$ is equationally Noetherian there exists a finite subsystem $S_0 \subseteq S$ such that $\mathrm{V}_{\mathcal{B}}(S_0) = \mathrm{V}_{\mathcal{B}}(S)$. Therefore, the following universal statement holds in $\mathcal{B}$

$$\forall\, y_1 \ldots \forall\, y_n \bigwedge_{(t=s)\in S_0} t(\bar{y}) = s(\bar{y}) \;\to\; \bigvee_{i=1}^{m} t_i(\bar{x}) = s_i(\bar{y}).$$

On the other hand the formula

$$\bigwedge_{(t=s)\in S_0} t(\bar{y}) = s(\bar{y}) \;\to\; \bigvee_{i=1}^{m} t_i(\bar{x}) = s_i(\bar{y})$$

is false in $\mathcal{C}$ under the interpretation $y_i \mapsto x_i$, $i = 1, \ldots, n$, hence $\mathcal{C} \notin \mathbf{Ucl}(\mathcal{B})$.

Equivalence 4) $\Longleftrightarrow$ 7) follows from Theorem 4.15.

□

Remark 6.1. In the case when $\mathcal{A} = \mathcal{B}$ the first two items in Theorem B can be formulated in a more precise form: $\mathcal{C} \equiv_{\forall,\mathcal{A}} \mathcal{A}$, and $\mathcal{C} \equiv_{\exists,\mathcal{A}} \mathcal{A}$, correspondingly.

References

[1] K. I. Appel, *One-variable equations in free groups*, Proc. Amer. Math. Soc., **19** (1968), pp. 912–918.

[2] B. Baumslag, *Residually free groups*, Proc. London Math. Soc., **17 (3)** (1967), pp. 402–418.

[3] G. Baumslag, *On generalized free products*, Math. Zeit., **7 (8)** (1962), pp. 423–438.

[4] G. Baumslag, A. Myasnikov, V. Remeslennikov, *Algebraic geometry over groups I: Algebraic sets and ideal theory*, J. Algebra, **219** (1999), pp. 16–79.

[5] G. Baumslag, A. Myasnikov, V. Remeslennikov, *Discriminating and co-discriminating groups*, J. Group Theory, **3 (4)** (2000), pp. 467–479.

[6] G. Baumslag, A. Myasnikov, V. Remeslennikov, *Discriminating completions of hyperbolic groups*, Geometriae Dedicata, **92** (2003), pp. 115–143.

[7] G. Baumslag, A. Myasnikov, V. Romankov, *Two theorems about equationally Noetherian groups*, J. Algebra, **194** (1997), pp. 654–664.

[8] M. Bestvina, M. Feighn, *Stable actions of groups on real trees*, Invent. Math. J., **121 (2)** (1995), pp. 287–321.

[9] R. Bryant, *The verbal topology of a group*, J. Algebra, **48** (1977), pp. 340–346.

[10] C. Champetier, V. Guirardel, *Limit groups as limits of free groups: Compactifying the set of free groups*, Israel J. Math., **146** (2005), pp. 1–76.

[11] O. Chapuis, *$\forall$-free metabelian groups*, J. Symbolic Logic, **62** (1997), pp. 159–174.

[12] D. Gaboriau, G. Levitt, F. Paulin, *Pseudogroups of isometries of R and Rips' theorem on free actions on R-trees*, Israel J. Math., **87** (1994), pp. 403–428.

[13] A. Gaglione, D. Spellman, *Some model theory of free groups and free algebras*, Houston J. Math., **19** (1993), pp. 327–356.

[14] V. Guirardel, *Limit groups and group acting freely on $\mathbb{R}^n$-trees*, Geometry and Topology, **8** (2004), pp. 1427–1470.

[15] V. A. Gorbunov, *Algebraic theory of quasivarieties*, Nauchnaya Kniga, Novosibirsk, 1999; English transl., Plenum, 1998.

[16] R. I. Grigorchuk, P. F. Kurchanov, *On quadratic equations in free groups*, Contemp. Math., **131(1)** (1992), pp. 159–171.

[17] D. Groves, *Limits of (certain) CAT(0) groups, I: Compactification*, Algebraic and Geometric Topology, **5** (2005), pp. 1325–1364.

[18] D. Groves, *Limit groups for relatively hyperbolic groups, II: Makanin-Razborov diagrams*, Geometry and Topology, **9** (2005), pp. 2319–2358.

[19] E. Daniyarova, *Foundations of algebraic geometry over Lie algebras*, Herald of Omsk University, Combinatorical methods in algebra and logic (2007), pp. 8–39.

[20] E. Daniyarova, I. Kazachkov, V. Remeslennikov, *Algebraic geometry over free metabelian Lie algebras I: U-algberas and universal classes*, J. Math. Sci., **135(5)** (2006), pp. 3292–3310.

[21] E. Daniyarova, I. Kazachkov, V. Remeslennikov, *Algebraic geometry over free metabelian Lie algebras II: Finite fields case*, J. Math. Sci., **135(5)** (2006), pp. 3311–3326.

[22] E. Daniyarova, *Algebraic geometry over free metabelian Lie algebras III: Q-algebras and the coordinate algebras of algebraic sets*, Preprint, Omsk, OMGU, 2005, pp. 1–130.

[23] E. Daniyarova, V. Remeslennikov, *Bounded algebraic geometry over free Lie algebras*, Algebra and Logic, **44(3)** (2005), pp. 148–167.

[24] O. Kharlampovich, A. Myasnikov, *Irreducible affine varieties over free group I: Irreducibility of quadratic equations and Nullstellensatz*, J. Algebra, **200 (2)** (1998), pp. 472–516.

[25] O. Kharlampovich, A. Myasnikov, *Irreducible affine varieties over free group II: Systems in trangular quasi-quadratic form and description of residually free groups*, J. Algebra, **200(2)** (1998), pp. 517–570.

[26] O. Kharlampovich, A. Myasnikov, *Algebraic geometry over free groups: Lifting solutions into generic points*, Contemp. Math., **378** (2005), pp. 213–318.

[27] O. Kharlampovich, A. Myasnikov, *Elementary theory of free nonabelian groups*, J. Algebra, **302 (2)** (2006), pp. 451–552.

[28] R. C. Lyndon, *Groups with parametric exponents*, Trans. Amer. Math. Soc., **96** (1960), pp. 518–533.

[29] G. Makanin, *Equations in free groups*, Izvestia AN USSR, math., **46(6)** (1982), pp. 1199–1273.

[30] A. I. Malcev, *Algebraic structures*, Nauka, Moscow, 1970.

[31] A. I. Malcev, *Some remarks on quasi-varieties of algebraic structures*, Algebra and Logic, **5 (3)** (1966), pp. 3–9.

[32] D. Marker, *Model theory: An introduction*, Springer-Verlag New York, 2002.

[33] A. Myasnikov, V. Remeslennikov, *Exponential groups 2: Extension of centralizers and tensor completion of CSA-groups*, International J. Algebra and Computation, **6(6)** (1996), pp. 687–711.

[34] A. Myasnikov, V. Remeslennikov, *Algebraic geometry over groups II: Logical foundations*, J. Algebra, **234** (2000), pp. 225–276.

[35] A. Myasnikov, V. Remeslennikov, D. Serbin, *Regular free length functions on Lyndon's free* $\mathbb{Z}(t)$*-group* $F^{\mathbb{Z}(t)}$, Contemp. Math., **378** (2005), pp. 37–77.

[36] B. Plotkin, *Varieties of algebras and algebraic varieties. Categories of algebraic varieties*, Siberian Advances in Math., **7 (2)** (1997), pp. 64–97.

[37] B. Plotkin, *Varieties of algebras and algebraic varieties*, Izrael J. Math., **96 (2)** (1996), pp. 511–522.

[38] A. Razborov, *On systems of equations in a free groups*, Combinatorial and geometric group theory, Edinburgh (1993), Cambridge University Press (1995), pp. 269–283.

[39] A. Razborov, *On systems of equations in a free groups*, Izvestia AN USSR, math., **48(4)** (1982), pp. 779–832.

[40] V. Remeslennikov, $\exists$*-free groups*, Siberian Math. J., **30(6)** (1989), pp. 998–1001.

[41] V. Remeslennikov, *Dimension of algebraic sets in free metabelian groups*, Fundam. and Applied Math., **7** (2000), pp. 873–885.

[42] V. Remeslennikov, R. Stöhr, *On algebraic sets over metabelian groups*, J. Group Theory, **8** (2005), pp. 491–513.

[43] V. Remeslennikov, R. Stöhr, *On the quasivariety generated by a non-cyclic free metabelian group*, Algebra Colloq., **11** (2004), pp. 191–214.

[44] V. Remeslennikov, N. Romanovskii, *Metabelian products of groups*, Agebra and Logic, **43(3)** (2004), pp. 190–197.

[45] V. Remeslennikov, N. Romanovskii, *Irreducible algebraic sets in metabelian groups*, Agebra and Logic, **44(5)** (2005), pp. 336-347.

[46] V. Remeslennikov, E. Timoshenko, *On topological dimension of u-groups*, Siberian Math. J., **47(2)** (2006), pp. 341-354.

[47] Z. Sela, *Diophantine geometry over groups I: Makanin-Razborov diagrams*, Publications Mathematiques de l'IHES, **93** (2001), pp. 31–105.

[48] Z. Sela, *Diophantine geometry over groups VI: The elementary theory of a free group*, GAFA, **16** (2006), pp. 707–730.